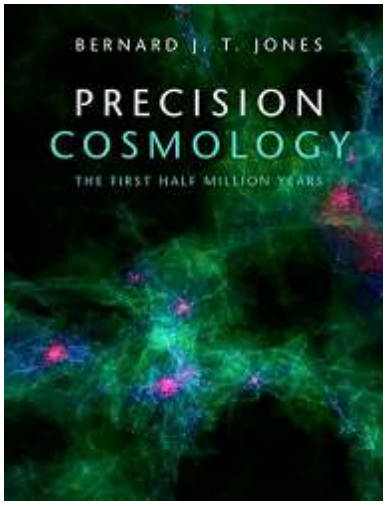
Lagrangians and Hamiltonians

A Supplement to “Precision Cosmology”

Bernard Jones



The 19^{th} century reformulations of classical mechanics by Lagrange and Hamilton opened up a deeper understanding of the nature of dynamical systems. That in turn allowed generalisations of the Lagrange and Hamilton approaches to descriptions of continuum mechanics, as in fluids, and then to quantum theory and the theory of quantum fields. These formulations of dynamics allowed discussions of the energy momentum tensor and led on to the study of symmetries in dynamical systems and in field theory.

This is one of a set of Supplementary Notes and Chapters to “Precision Cosmology”.
Some of these Supplements might have been a chapter in the book itself, but were regarded either as being somewhat more specialised than the material elsewhere in the book, or somewhat tangential to the main subject matter.
The are mostly early drafts and have not been fully proof-read.
Please send comments on errors or ambiguities to “PrecisionCosmology(at)gmail.com”.

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1.1 Generalised coordinates

At the heart of this formalism of dynamics lies the *Lagrangian* for the physical system, $\mathcal{L}$ which is an explicit function of the what are known as *generalised coordinates*, $\mathbf{q}$, and their time derivatives, $\dot{\mathbf{q}}$. In classical mechanics we usually think of the $\mathbf{q}$ as describing the configuration of the system and the $\dot{\mathbf{q}}$ as being the time derivatives of those descriptors. The space of $\mathbf{q}$-values is referred to as the *configuration space* of the system. The number of elements $\mathbf{q} : \{q_1, \ldots, q_n\}$ needed to describe the configuration of the system is referred to as the dimension, n, of the configuration space. [1] We can display the explicit dependence of L on the variables $\mathbf{q}$ by writing

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1.1}$$

When there is no explicit time dependence, the energy of the system is conserved.

1.2 Equations of motion

The equations of motion are derived from a quantity called the "action", $\mathcal{S}$, which is given in terms of the Lagrangian L by [2]

$$\mathcal{S} = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \tag{1.2}$$

The value of this integral, $\mathcal{S}$ clearly should depend on the path in $\mathbf{q}$-space between the end-points, t_1 and t_2. The dynamical equations are then derived from an *action principle* which states that the path taken by the system in $(\mathbf{q}, \dot{\mathbf{q}})$-space is the path which minimises

[1] The Hamiltonian view uses a $2n$-dimensional *phase space* built up from the generalised coordinates q^a and their *conjugate momenta*, p^a.

[2] We put aside here the problem of describing possible constraints $f_a(\mathbf{q}, t) = 0, a = 1, \ldots, m$ on the $\mathbf{q}$. These have to be added to the Lagrangian in equation (1.2) using *Lagrange multipliers*, $\lambda_a, a = 1, \ldots, m$, one for each constraint. The λ_a take on the role of additional dynamical variables that have to be eliminated from the resulting equations.

the action integral $\mathcal{S}$:

$$\delta\mathcal{S}[\mathbf{q}] = \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt = 0 \tag{1.3}$$

The $\delta\mathcal{S}$ operation is to be thought of as a variation, relative to the true path, of all the paths that the system could take between the times t_1 and t_2. This is referred to as *Hamilton's Principle* and is often written as a functional derivative:

$$\frac{\delta\mathcal{S}}{\delta\mathbf{q}(t)} = 0 \tag{1.4}$$

The integral for $\mathcal{S}$ depends on the path taken between the end points t_1, t_2 and so $\mathcal{S}$ is referred to as a *functional* of the path. The functional derivative used in (1.4) represents the variation in the value of the integral among neighbouring paths. The paths that minimise this integral are of special importance in physics (Feynman and Hibbs, 1965).

Performing this variation is a standard part of the calculus of variations and yields the *Lagrange equations* for the motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = 0, \quad a = 1, \ldots, n. \tag{1.5}$$

where n is the number of variables in the vector $\mathbf{q}$ (the dimension of the configuration space of the system). The quantity

$$p_a = \frac{\partial L}{\partial q^a} \tag{1.6}$$

is the *conjugate momentum* for the coordinate q^a which allows us to define the *Hamiltonian* of the system

$$H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p} \cdot \mathbf{q} - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1.7}$$

Note that the Hamiltonian is now a function of the $2n$ variables q^a and p_a and time, t. Hamilton's equations of motion are then

$$\dot{q}^a = \frac{\partial H}{\partial p^a}, \quad -\dot{p}_a = \frac{\partial H}{\partial q^a}, \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{1.8}$$

1.3 Conservative systems

In classical mechanics we generally write the Lagrangian as the difference between the kinetic energy T and potential energy V expressed as functions of the $(\mathbf{q}, \dot{\mathbf{q}})$:

$$L = T - V \tag{1.9}$$

This is a good working model for most dynamical systems, but the Lagrangian does not always have to be of this form. For conservative systems, where the Lagrangian does not depend explicitly on time, the Hamiltonian is simply

$$H = T + V \tag{1.10}$$

though this has to be expressed in terms of the q^a's and their conjugate momenta p_a, which involves differentiating the Lagrangian unless one has some independent means of knowing what the p_a are.

1.4 Symmetries

If the Lagrangian does not depend explicitly on one of the coordinates q^m then we see from the Lagrange equation (1.5) that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = 0 \quad \text{for } m \text{ such that } \frac{\partial L}{\partial q^m} = 0. \tag{1.11}$$

Hence

$$\frac{\partial L}{\partial \dot{q}^m} = \text{constant for } m \text{ such that } \frac{\partial L}{\partial q^m} = 0. \tag{1.12}$$

So whenever a coordinate such as q_m does not appear specifically in the Lagrangian, there is an associated conservation law. What is important is that we can transform q^m in any way we want and the associated conservation law still holds. Such a coordinate is said to be an *ignorable coordinate*, or *cyclic coordinate*, and reflects a symmetry of the system. There was a substantial amount of work in the 19^{th} century trying to manipulate coordinate systems so that one or more variables would become ignorable: each time this could be done, the differential equation corresponding to that variable was instantly integrated. [3]

[3] There is another scaling symmetry that occurs when the potential term $V(\mathbf{q})$ is a homogeneous function of the q^i, *i.e.* $V(\alpha\mathbf{q}) = \alpha^k V(\mathbf{q})$ for some non-zero k, not necessarily real.

References

Feynman, R.P., and Hibbs, A.R. 1965. *Quantum Mechanics and Path Integrals.* McGraw-Hill.