# Functions on a Sphere

A Supplement to “Precision Cosmology”

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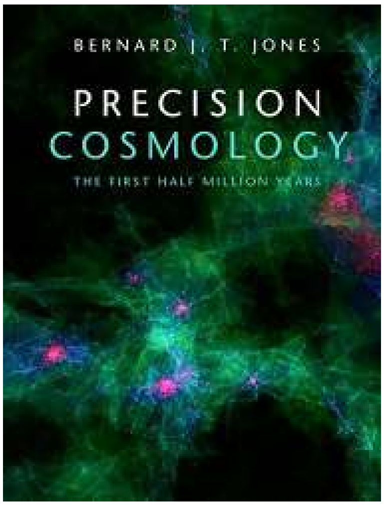


This is an overview of the analysis of scalar, vector and tensor fields on a sphere using various forms of spherical harmonics. It is shown how to apply this to the CMB temperature and polarisation data.

This is one of a set of Supplementary Notes and Chapters to “Precision Cosmology”.
Some of these Supplements might have been a chapter in the book itself, but were regarded either as being somewhat more specialised than the material elsewhere in the book, or somewhat tangential to the main subject matter.
The are mostly early drafts and have not been fully proof-read.
Please send comments on errors or ambiguities to “`PrecisionCosmology(at)gmail.com`”.

# Contents

# 1 Functions on a Sphere

## 1.1 Multipole expansions

We observe the universe from only one point of view: the Earth and its immediate space environment. What this means is that we observe the universe only in projection on the celestial sphere. It should be no surprise that representing observational data involves the use of spherical polar coordinates and much of the mathematics that comes along with that. So, just as working in Cartesian coordinates we might represent a density field in terms of its Fourier components, the would represent a field on the surface of the celestial sphere in terms of the analogous Spherical harmonic functions. Because of the inherent complexity of spherical coordinates that arises from the need to use trigonometric functions, we would expect the descriptions of physical quantities expressed in spherical coordinates to likewise be more complex, or even intimidating.

The question naturally arises as to why we should buy this extra complexity: what do we get by investing in this enormous and technically difficult machinery? A part of the answer is because the spherical harmonic functions are a set of orthogonal functions on the sphere, and so allow the decomposition of a function defined on the sphere to be split into independent components. Moreover, those independent components carry with them the notion of frequency: the components describe the content of the function at different levels of resolution. This opens up the possibility of smoothing the function in a resolution dependent manner. The spherical harmonics, being the spherical analogue of the Fourier transform, provide definitions in spherical coordinates of the power spectrum of a field and its generalisations.[1]

Although the use of spherical harmonics has a long history, particularly in electromagnetism and in quantum mechanics, their use in cosmology only goes back to the early work of Peebles and his group in the 1970's. His book, Peebles (1980, Ch. III, especially §46 *et seq.* ), is an essential reference on the analysis of data on a sphere.

Here we discuss the representation of data on a sphere, or, more generally, in spherical polar coordinates. There are two issues to tackle: the representation of scalar data, like a scalar potential, and the representation of vector data, like an electric field. While much of the underlying mathematical work was developed in the $19^{th}$ and first half of the $20^{th}$ centuries with the evolution of potential theory, the recent developments have come from

[1] There are alternative representation of data on a sphere, one of the most important being the spherical wavelet representation (Barreiro et al., 2000, Appendix A presents spherical Haar wavelets). Smoothing data on a sphere, in a way that is analogous to using a Gaussian window function to smooth data on a plane, can be achieved by use of the spherical analogues of the Gaussian: the univariate von Mises-Fisher distribution (Section **??**) and the bivariate Kent distribution (Section **??**).

atomic physics and quantum fields where there is a need to represent wave functions in terms of angular momentum operators, *e.g.* Condon and Shortley (1935). The need to describe tensor functions on a sphere has arisen both in general relativity, particularly with regard to multipole expansions of gravitational radiation (Regge and Wheeler, 1957; Zerilli, 1970), and in geophysics, in connection with describing stress tensors and seismic phenomena (Burridge, 1969).

The vector and tensor spherical harmonics can be constructed by acting on the better-known scalar spherical harmonics, $Y_{lm}$, with certain first and second order differential operators. This approach reflects the notion that, although the components of vectors and tensors are not in fact scalars, it should somehow be possible to use the scalar spherical harmonics to describe them. The trick is to use the directional derivatives of the $Y_{lm}$, although this is easier said than done.

One of the outstanding issues here is the matter of conventions: different disciplines use different sign conventions, and variations of convention within a discipline are not infrequent. Authors rarely state which conventions are being used. Here we follow the notation and conventions adopted by Jackson (1998, §§3.4, 16.8) and Arfken and Weber (2005, §§12.6, 12.11). For the tensor spherical harmonics and a more mathematical approach see Backus (1967). From a quantum mechanics point of view see Edmonds (1996, particularly §5.9) and Thompson (1994, Ch. 4.).

## 1.2 Representing data on a sphere

Fourier analysis of data defined in a rectangular box using Cartesian coordinates exploits the trigonometric functions sin and cos. We need to define an analogous set of functions that allow the analysis of data defined in a sphere using spherical polar coordinates. Given the ubiquity of the Laplacian operator $\nabla^2$ in physics (*e.g.* gravitation, electromagnetism, quantum mechanics, etc.), a good starting place is to look at solutions of the Laplace equation $\nabla^2 u = 0$ expressed in spherical polar coordinates $(r, \theta, \phi)$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0 \qquad (1.1)$$

This is arguably the most widely applicable equation in all of physics. Traditionally, this is solved by the method of separation of variables wherein we postulate the existence of a solution of the form $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ that satisfies the boundary conditions of the problem.[2]

The separation of variables yields ordinary second order differential equations for the functions $R(r), \Theta(\theta), \Phi(\phi)$, and so the boundary conditions are expected to provide values for the two constants of integration that are required for the solution each equation. The

[2] An obvious boundary condition might be that the solution be finite and continuous and differentiable everywhere. Some problems require that the solution vanishes at infinity and further impose conditions of spherical or axial symmetry. More general boundary conditions might specify the function or its gradient on a given surface.

process is described in most textbooks on modern physics and mathematics so we only need the results here.

The radial equation for $R(r)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda \tag{1.2}$$

where $\lambda$ is a constant of separation, which at this stage of the process is arbitrary and can, with hindsight, be written as $\lambda = l(l+1)$ for some arbitrary number $l$, which at this stage is not necessarily an integer. After making a transformation of variables $r = e^t$ the solution is simple and when expressed in terms of the original $r$-variable the radial solution is

$$R(r) = Ar^l + Br^{-(l+1)} \tag{1.3}$$

for constants of integration $A, B$ that are determined by radial boundary conditions. Already we see that there is no $A, B$ that can satisfy 'finite everywhere' and 'vanishes at infinity' unless both are zero.

The next step of the separation is to separate out the $\phi$ dependence and we get the agreeably simply equation

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m^2\,\Phi \tag{1.4}$$

where $-m^2$ is another separation constant, which, again with hindsight, we take to be negative. This provides the solution

$$\Phi(\phi) = C\sin\phi + D\cos\phi \tag{1.5}$$

Had we chosen a positive separation constant $+m^2$ we would have had hyperbolic functions which would not have allowed continuity and differentiability in $\phi$ at $\phi = 0$ and $\phi = 2\pi$. The sin and cos solution allow this if $m$ is a non-zero integer.

This leads us finally to the $\Theta(\theta)$ equation, which, with $\mu = \cos\theta$ is:

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dy}{d\mu}\right) + \left(l(l+1) - \frac{m^2}{1-\mu^2}\right)y = 0 \tag{1.6}$$

where $y(\mu)$ is $\Theta(\theta)$. This is the associated Legendre equation whose solutions we shall analyse in depth later.

Solving the Laplace equation in this way leads to special functions that are associated with that equation. If we had been solving the equation $\nabla^2 u + k^2 u = 0$ we would have ended up with different radial functions, the spherical Bessel functions.

It should be stressed that there is no unique representation of data in spherical polar coordinates, nor for that matter, in any coordinate system. We could, for example, use radial basis functions for describing the radial distribution of data, or we could use wavelets to describe the data on spherical surfaces. The answer to the often asked question "why are we using these weird functions?" is that they may simply be more convenient than many alternatives. However, if there were a physical basis for using a particular set of functions, as in the case of describing electromagnetic phenomena with Legendre functions and their relatives, there would need to be a special reason to use anything else.[3]

[3] There are several important papers using other mechanisms, including wavelets, to analyse the temperature

### 1.2.1 Legendre polynomials

The starting point is to look at axially symmetric solutions, and thus eliminate the $\phi$-dependence and the function $\Phi(\phi)$. This corresponds to equation (1.6) with $m = 0$ which we can write, with the replacement $x \leftrightarrow \mu$, as

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + l(l+1)P = 0. \tag{1.7}$$

For every integer values of $l$ this has a polynomial solution on the interval $[-1, 1]$ denoted by $P_l(x)$.[4] The solutions can be expressed through *Rodriguez's formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \geq 0, \tag{1.8}$$

which is also used as a definition of the polynomials. The first few of these, writing $x = \cos\theta$, are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= \cos\theta \\ P_2(x) &= (3\cos^2\theta - 1)/2 \\ P_3(x) &= (5\cos^3\theta - 3\cos\theta)/2 \\ &\dots \end{aligned} \tag{1.9}$$

where $x = \cos\theta$. The angle $\theta$ is the angle around the circle. A simple way to generate the $P_l(x)$ is to use the recursion formula

$$(n+1)P_n(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0(x) = 1$$

### 1.2.2 Associated Legendre functions

The *Associated Legendre Functions* are the solutions of the associated Legendre equation

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0, \tag{1.10}$$
$$l = 0, 1, 2, ..., \quad \text{and} \quad m = 0, \pm 1, \pm 2, \dots, \pm l$$

which arises in the solution of Laplace's equation in spherical polar coordinates by the separation of variables. The solutions are denoted by $P_l^m(x)$ and clearly $P_l^0(x) = P_l(x)$.

The $P_l(x)$ are solutions for $m = 0$. Values of $m \neq 0$ give rise to the *Associated Legendre Functions of the first kind*, $P_l^m(x)$, $l = 0, 1, 2, \dots$ with integer $m$ in the range $-l \leq m \leq l$. These functions are regular at $x = \pm 1$ and are discussed below, see equation (1.21).

distribution of the cosmic background radiation. **?** introduce windowed spherical harmonics akin to the Gabor transform in plane geometry and calculate the associated pseudo power spectra.

[4] Equation (1.7) has polynomial solutions, denoted by $P_l(x)$, and non-polynomial solutions, denoted by $Q_l(x)$. The $P_l(x)$ are *Legendre functions of the first kind*, while the $Q_l(x)$ are *Legendre functions of the second kind*.

The functions $P_n^m(x)$ can be written as[5]

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad m \ge 0 \tag{1.11}$$

$$= (-1)^m \frac{1}{2^n n!} (1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n, \quad -l \le m \le l, \tag{1.12}$$

and satisfy the the orthogonality relations

$$\int_{-1}^{+1} P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} = 0, \quad k \ne m. \tag{1.13}$$

$$\int_{-1}^{+1} P_n^m(x) P_k^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{km} \tag{1.14}$$

It is easy to verify the relationship

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{1.15}$$

Press et al. (2007, §6.8) recommends the recurrence relationship on $l$

$$(l-m) P_l^m(x) = x(2l-1) P_{l-1}^m(x) - (l-m+1) P_{l-2}^m(x) \tag{1.16}$$

$$P_m^m = (-1)^m (2m-1)!!(1-x^2)^{m/2}, \quad P_{m+1}^m = x(2m+1) P_m^m, \tag{1.17}$$

as a stable way of generating the $P_l^m(x)$.[6]

When we put $\cos\theta = x$, the $P_l^m(\cos\theta)$ are referred to as the *Associated Legendre Functions*, the first few of which are

$$\begin{aligned} &P_0^0(\cos\theta) = 1 && P_2^0(\cos\theta) = (3\cos^2\theta - 1)/2, \\ &P_1^0(\cos\theta) = \cos\theta, && P_2^1(\cos\theta) = -3\cos\theta\sin\theta, \\ &P_1^1(\cos\theta) = -\sin\theta && P_2^2(\cos\theta) = 3\sin^2\theta \\ &\quad \ldots \end{aligned} \tag{1.18}$$

## 1.3 Spherical harmonics

The *spherical harmonics* are a set of orthonormal basis functions on the Euclidean 3-sphere. The qualification that it should be a 'Euclidean 3-sphere' is not often made in physics, but it is important when studying physical phenomena in curved space-times. They are the spherical analogue of the simpler Fourier basis functions on a plane rectangle having periodic boundary conditions.

It is worth reminding ourselves how the vibrations of a rectangular plate is handled. The basic equation for the vibration amplitude $h(x, y)$ at a point $(x, y)$ of the plate is the two-dimensional wave equation $(\partial_x^2 + \partial_y^2)\, h(x, y) = -k^2 h(x, y)$, where $k = \omega/c$ for waves

[5] These equations, known as the *general Rodriguez formulae*, are often taken as the definition of the $P_l^m(x)$ since they specify the normalisation and sign of the functions. Note that this expression for $P_l^m(x)$ is sometimes given without the $(-1)^m$ factor, in which case the $(-1)^m$ reappears in the definition (1.21) of the $Y_{lm}$ .

[6] $n!!$ is the product of all odd integers $\le n$.

of temporal frequency $\omega$ travelling with velocity $c$. This is the Helmholtz equation, usually written as $(\nabla^2_{(2)} + k^2)h(x,y) = 0$, where $\nabla^2_{(2)}$ denotes the Laplacian operator in two dimensions.

If we look for separable solutions and write $h(x,y) = X(x)Y(y)$, we find that $X(x)$ and $Y(y)$ satisfy $X(x) = X_0 e^{imx}$ and $Y(y) = Y_0 e^{iny}$, for constants $m, n$ such that $m^2 + n^2 = k^2$ and where $X_0, Y_0$ are complex constant amplitudes. The details of the solution are fixed by the boundary conditions which could be periodic boundary conditions, or $h(x,y) = 0$ on the boundary of the plate.

The situation with regard to describing functions $h(\theta,\phi)$ on a sphere is not very different. The basis functions on a sphere are denoted by $Y_{lm}(\theta,\phi)$ and they are solutions of the inhomogeneous Poisson equation

**The $Y_{lm}$ as eigenfunctions**

$$\nabla^2 Y_{lm} = -\frac{l(l+1)}{r^2} Y_{lm} \tag{1.19}$$

which is a Helmholtz equation in which the frequency depends on the radius $r$ of the sphere. The $Y_{lm}$ form a complete set of orthogonal functions from which general solutions can be built.[7]

### 1.3.1 $Y_{lm}$: the basis functions

Using the $P_l^m(x)$ we can define a set of orthogonal polynomials, called *Spherical Harmonics*, which are defined over the surface of a sphere. The spherical harmonics are expressed in terms of the familiar spherical polar angles $(\theta,\phi)$, $-\pi \le \theta \le \pi,\ 0 \le \phi < 2\pi$:

**Definition 1.1**
**Spherical harmonic basis functions:**

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi},\ \ -l \le m \le l \tag{1.21}$$

[7] We stressed earlier that this analysis applied to a 2-sphere in a Euclidean 3-space. When we can describe a curved 3-space in terms of a line element in the form $dl^2 = dr^2 + \sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ we can take $\sigma(r) = \sin r, r, \sinh r$ according as to whether the space has negative, zero or positive curvature. The Helmholtz equation then takes the form

$$\nabla^2 Y_{lm} = -\frac{l(l+1)}{\sigma(r)^2} Y_{lm} \tag{1.20}$$

Hence working in curved space and using this form of the metric ensures that the appropriate spherical basis functions are still the $Y_{lm}$: only the radial basis functions are affected.

The $Y_{l,m}(\theta,\phi)$ are called *spherical harmonics*[8]. The normalisation factor comes from equation (1.14).[9]

The relationship between $Y_{lm}$ and $Y_{l,-m}$ is

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi) \tag{1.22}$$

where $Y^*_{lm}$ denotes the complex conjugate of $Y_{lm}$ (note equation 1.15). The orthogonality condition is expressed as

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y^*_{l_1,m_1}(\theta,\phi)Y_{l_2,m_2}(\theta,\phi)\;\sin\theta d\theta\, d\phi = \delta_{l_1 l_2}\delta_{m_1 m_2}. \tag{1.23}$$

(see Arfken and Weber (2005, eq. 12.154)).

The first few $Y_{lm}$ are

***monopole***: $l = 0$

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}$$

***dipole***: $l = 1$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

***quadrupole***: $l = 2$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

$$Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi} \tag{1.24}$$

The importance of these functions is that they form a set of orthogonal functions on a sphere, and so any (bounded) function $f(\theta,\phi)$ on the sphere can be represented as a linear sum of $Y_l^m$'s:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi) \tag{1.25}$$

for a set of constant $a_{lm}$ which are determined by the equations

$$a_{lm} = \int_{-1}^{1}\int_{0}^{2\pi} Y^*_{lm}(\theta,\phi)f(\theta,\phi)\,d\phi\; d(\cos\theta) \tag{1.26}$$

[8] Also referred to as *surface harmonics of the first kind*. For $m < n$ they are referred to as *tesseral harmonics* and for $m = n$ as *zonal harmonics*. Sometimes this definition is given with an additional pre-multiplying factor of $(-1)^m$, but in most such cases the definition (1.12) of the $P_l^m$ comes without the $(-1)^m$ pre-multiplier, as is the case in Arfken and Weber (2005, eq. 12.153). Our convention follows, for example, Press et al. (2007, §6.8) and Jackson (1998, §3.4).

[9] Note that the normalisation factor shown here is not necessarily the same as used in other areas of analysis. The present variant is the *Condon Shortley phase* choice and is responsible of the $(-1)^m$ in equation (1.21). An alternative convention is the *Darwin phase* choice which is distinguished by having $|m|$ in place of $m$ in equation (1.21) and no factor of $(-1)^m$.

This is completely analogous to the Fourier representation of a function.

### 1.3.2 Sum rules and the addition theorem

For $m = 0$:

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{1.27}$$

The sum of the squares of the $Y_{lm}$ satisfies the "sum rule"

**Sum rule**

$$\sum_{m=-l}^{l} |Y_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi} \tag{1.28}$$

Finally there is the important "addition theorem" for spherical harmonics: this expresses the sum of products of Legendre functions for two directions $(\theta, \phi)$, $(\theta', \phi')$ in terms of the angle $\gamma$ between these directions as

**Addition theorem**

$$\frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi) Y_{lm}(\theta', \phi') = P_l(\cos\gamma) \tag{1.29}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta' \cos(\phi - \phi') \tag{1.30}$$

The limit $\gamma \to 0$ yields the sum rule (1.28).

### 1.3.3 Combinations of three $Y_{lm}$'s

There is an important relationship involving the integral of three spherical harmonics (Arfken and Weber, 2005, §12.9):

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y^*_{l_1 m_1}(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) Y_{l_3 m_3}(\theta, \phi) \; \sin\theta d\theta d\phi = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{1.31}$$

The symbol $\left(\begin{smallmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ is a *Wigner* $3j$*-symbol*. These can be calculated via the *Racah formula*, see Messiah (1962, p. 1058) who provides a short table on *loc. cit.* p.1060.

### 1.3.4 Relationship between $Y_{lm}$ and Fourier components

Legendre functions provide but one way of decomposing a field into orthogonal functions, and are particularly useful when discussing the spherical distribution of physical quantities about a given point. The more usual way of representing the spatial distribution of a function is to use the Fourier decomposition of the field. The Fourier and spherical harmonic representations of a function are related simply by knowing how to express plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ in Legendre polynomials $P_l(\cos\gamma)$:

**Plane wave expressed in Legendre polynomials**

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr) P_l(\cos\gamma) \tag{1.32}$$

where $\gamma$ is the angle between $\mathbf{k}$ and $\mathbf{r}$ (*i.e.* $\cos\gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{r}}$, where $\hat{\mathbf{r}} = \mathbf{r}/|\,\mathbf{r}\,|$, $\hat{\mathbf{k}} = \mathbf{k}/|\,\mathbf{k}\,|$).[10] The function $j_l(kr)$ is a *spherical Bessel function of the first kind* of order $l$.[11]

There is no closed-form expression for $j_l(x)$. It is one of the family of solutions of the differential equation

$$x^2 y'' + 2xy' + [x^2 - l(l+1)]y = 0, \quad l = 0, \pm 1, \pm 2, \ldots \tag{1.35}$$

and is related to the Bessel functions $J_{l+\frac{1}{2}}(x)$ through

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x) \tag{1.36}$$

See, for example, Press et al. (2007, §6.7 eq. 6.7.47).

### 1.3.5 Relating spherical and Cartesian Fourier transforms

If the function $f(\mathbf{r})$, is defined in 3-volume and has Fourier transform $g(\mathbf{k})$ then, with our normalisation of the Fourier Transform (see Appendix **??**, section **??**):

$$f(\mathbf{r}) = \int_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} \tag{1.37}$$

Note that this equation, and hence the subsequent analysis, applies only to a function $f(\mathbf{r})$ defined on a flat space in which the functions $e^{i\mathbf{k}\cdot\mathbf{r}}$ form a complete orthonormal basis on the Cartesian frame of reference $\{x_i\}$.

10 This is proved by representing $e^{ikr\cos\theta}$ as a series in $P_l(\cos\theta)$: $e^{ikr\cos\theta} = \sum_{l=0}^{\infty} B_l(r)\, P_l(\cos\theta)$, and determining the coefficients $B_l(r)$ using the orthogonality of the $P_l(\cos\theta)$. This gives us

$$B_l(r) = \frac{1}{2}(2l+1)\int_{-1}^{+1} e^{ikr\cos\theta} P_l(\cos\theta)\, d(\cos\theta) \tag{1.33}$$

The result $B_l(r) = (2l+1)\, i^l\, j_l(kr)$ follows from knowing that the value of the integral is $2i^l\, j_l(kr)$ (Morse and Feshbach, 1953, Part II, p.1575).

11 If we introduce the result of the addition theorem (1.30) into the plane wave expansion (1.32) we get the alternate expression

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} i^l\, j_l(kr) Y_{lm}(\hat{\mathbf{x}}) Y^*_{lm}(\hat{\mathbf{k}}) \tag{1.34}$$

where we have denoted the directions of the vectors $\mathbf{r}$ and $\mathbf{k}$ by the symbols $\hat{\mathbf{r}} = \mathbf{r}/|\,\mathbf{r}\,|$ and $\hat{\mathbf{k}} = \mathbf{k}/|\,\mathbf{k}\,|$.

If we now use equation (1.34) for $e^{i\mathbf{k}\cdot\mathbf{r}}$ we have

$$f(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} 4\pi \int_{\mathbf{k}} g(\mathbf{k})\, i^l Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{k}})d^3\mathbf{k} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_{lm}(\hat{\mathbf{r}}) \int_{\mathbf{k}} i^l j_l(kr)Y^*_{lm}(\hat{\mathbf{k}})g(\mathbf{k})\, d^3\mathbf{k} \tag{1.38}$$

where, as before, $\hat{\mathbf{r}} = \mathbf{r}/|\,\mathbf{r}\,|$ and $\hat{\mathbf{k}} = \mathbf{k}/|\,\mathbf{k}\,|$ are the directions of the vectors $\mathbf{r}$ in real space and $\mathbf{k}$ in $\mathbf{k}$-space.

If we now write the spherical harmonic transform of $f(\mathbf{r})$ as

$$f(\mathbf{r}) = f(r, \hat{\mathbf{r}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{lm}\, Y_{lm}(\hat{\mathbf{r}}) \tag{1.39}$$

and compare this with the previous equation, we get the important result that[12]

**Relating Fourier plane wave expansion to spherical harmonic expansion**

$$a_{lm} = 4\pi \int_{\mathbf{k}} i^l\, j_l(kr)Y^*_{lm}(\hat{\mathbf{k}})\, g(\mathbf{k})d^3\mathbf{k} \tag{1.40}$$

This equation will enable us to get the spherical harmonic representation, *i.e.* the coefficients $a_{lm}$, that represent a function whose Fourier transform is $g(\mathbf{k})$. In particular we shall apply this to the case when $g(\mathbf{k})$ is a random field with power spectrum $\mathcal{P}(k)$. If $g(\mathbf{k})$ is the distribution of cosmic density fluctuations, the $a_{lm}$ provide the spherical representation of that density field on a sphere of radius $r$ from which we can calculate the power spectrum of measurements of the field on the surface of the sphere (see section 1.9).

In the cosmological context of observing temperature fluctuations on the surface of last scattering of CMB photons, we have $r = 2cH_0^{-1}$.

### 1.3.6 Parseval's theorem in spherical coordinates

The equivalent of the Cartesian space Fourier transform expressed in spherical polar coordinates is

$$f(r, \theta, \phi) = \int_{R^3} F(k, \Theta, \Phi)e^{i\mathbf{k}\cdot\mathbf{r}}\, k^2 dk\, \sin(\Theta)d\Theta\, d\Phi \tag{1.41}$$

where $(r, \theta, \phi)$ are the usual real space spherical polar coordinates and $(k, \Theta, \Phi)$ are spherical polar coordinates in the Fourier transform space.

The mean square modulus, or total power, of the function $F(k, \Theta, \Phi)$ in the Fourier domain is

$$E_k = \int_{\mathbb{R}^3} |F(k, \Theta, \Phi)|^2 k^2 \sin(\Theta)\, dk\, d\Theta\, d\Phi \tag{1.42}$$

[12] This derivation is due to Rien van de Weygaert. The result seems to have first appeared in the cosmology literature, without being derived, in the paper of Peebles (1982, eq. 9). See also Abbott and Wise (1984b,a) and Kolb and Turner (1990, §9.6.2).

while the total power of its real space counterpart $f(r,\theta,\phi)$ is

$$E_r = \int_{\mathbb{R}^3} |f(r,\theta,\phi)|^2 r^2 \sin(\theta)\, dr\, d\theta\, d\phi \tag{1.43}$$

We then can use the expression 1.41 for $f(r,\theta,\phi)$ in the equation for $E_r$ to obtain

$$\begin{aligned} E_r = \int_{\mathbb{R}^3} dr\, d\theta\, d\phi \left[\int_{\mathbb{R}^3} F(k,\Theta,\Phi) e^{i\mathbf{k}\cdot\mathbf{r}} k^2 \sin(\Theta) dk\, d\Theta\, d\Phi\right] \\ \times \left[\int_{\mathbb{R}^3} F(k',\Theta',\Phi')^* e^{-i\mathbf{k}'\cdot\mathbf{r}} k'^2 \sin(\Theta') dk'\, d\Theta'\, d\Phi'\right] r^2 \sin(\theta) \end{aligned} \tag{1.44}$$

. Exchanging the order of integration and using the completeness relation

$$\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r = (2\pi)^3 \delta^3(\mathbf{k}) = \frac{\delta(k)\delta(\Theta)\delta(\Phi)}{k^2 \sin(\Theta)} \tag{1.45}$$

yields

$$E_r = (2\pi)^3 \int_{\mathbb{R}^3} |F(k,\Theta,\Phi)|^2 k^2 \sin(\Theta)\, dk\, d\Theta\, d\Phi = (2\pi)^3 E_k. \tag{1.46}$$

which is the spherical polar case of the familiar Parseval Relationship.

## 1.4 Representing a vector field on a sphere

We have seen how to represent a scalar function $f(\theta,\phi)$ defined on a sphere in terms of the spherical harmonics on a sphere:

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(r) Y_{lm}(\theta,\phi) \tag{1.47}$$

$$a_{lm}(r) = \int_{-1}^{1} \int_{0}^{2\pi} Y^*_{lm}(\theta,\phi) f(r,\theta,\phi)\, d\phi\, d(\cos\theta) \tag{1.48}$$

These are just equation (1.25) and (1.26), but with the radial dependence put back in so that we are not confined to the surface of a sphere.

We can likewise write the representation of a vector field $\mathbf{E}$ in spherical polar coordinates as

$$\mathbf{E}(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[a_{lm}\mathbf{L}(r,\theta,\phi) + b_{lm}(r)\mathbf{M}(r,\theta,\phi) + c_{lm}(r)\mathbf{N}(r,\theta,\phi)\right] \tag{1.49}$$

for a set of functions $a_{lm}, b_{lm}, c_{lm}$ and to-be-determined orthogonal function sets $\mathbf{L}(\theta,\phi)$, $\mathbf{M}(\theta,\phi)$ and $\mathbf{N}(\theta,\phi)$.[13] Equation (1.51) motivates the choice

$$\mathbf{Y}_{lm} = Y_{lm}\, \mathbf{e}_r, \quad \mathbf{\Psi}_{lm}(\theta,\phi) = r\boldsymbol{\nabla} Y_{lm}(\theta,\phi), \quad \mathbf{\Phi}_{lm} = \mathbf{r} \times \nabla Y_{lm}(\theta,\phi) \tag{1.52}$$

[13] In a fine didactic development of these functions, Barrera et al. (1985) provide the motivation that if we take the gradient of

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_{lm}(\theta,\phi) \tag{1.50}$$

So for each $l, m$ we have a triad of basis vectors $(\mathbf{Y}_{lm}, \mathbf{\Psi}_{lm}, \mathbf{\Phi})$, the *vector spherical harmonics* relative to which a field $\mathbf{E}$ can be expanded.

The basis provided by these functions is in fact complete. See also Jackson (1998, §16.2).[14] We note that, since $\mathbf{Y}_{lm}$ and $\mathbf{\Phi}_{lm}$, are mutually orthogonal, we could write the third component $\mathbf{\Psi}_{lm}(\theta,\phi)$ as $\mathbf{r} \times (\mathbf{e}_r \times \mathbf{\nabla}) Y_{lm}(\theta,\phi)$.

An important way of expressing the components (1.52) of the basis vector $(\mathbf{Y}_{lm}, \mathbf{\Psi}_{lm}, \mathbf{\Phi})$ is to write it as

$$(\mathbf{Y}_{lm}, \mathbf{\Psi}_{lm}, \mathbf{\Phi}) = \Big(\mathbf{e}_r,\ r\mathbf{\nabla},\ r\mathbf{e}_r \times \mathbf{\nabla}\Big)\, Y_{lm}(r,\theta,\phi). \tag{1.53}$$

which serves to emphasise the view of these basis vectors as differential operators.

### 1.4.1 Representing an electromagnetic wave

For an electromagnetic wave of frequency $\omega$ and wavenumber $k$, the electric field cab be written $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(r)e^{-i\omega t}$, with $k = \omega c$. We can then write the vacuum Maxwell equations as

$$(\nabla^2 + k^2)\mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0, \quad \mathbf{B} = -ik^{-1}\nabla \times \mathbf{E} \tag{1.54}$$

The solution is found by separation of variables. The angular dependence of the solution is expressed in terms of the vector spherical harmonics, and the $k^2$ term means that the radial solution is expressed in terms of the spherical Bessel functions (see Section 1.2). To represent a linearly polarised wave, we need two independent solutions like this.

Following on from this the representation of the electric field $\mathbf{E}(\mathbf{r})$ of a wave in terms of spherical harmonics can be written as

$$\mathbf{E}(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}\mathbf{M}_{lm}(r) + b_{lm}\mathbf{N}_{lm}(r)\right] \tag{1.55}$$

where $\mathbf{M}_{lm}(\mathbf{r})$ and $\mathbf{N}_{lm}(\mathbf{r})$ are spherical wave functions. These are given by

$$\mathbf{M}_{lm}(\mathbf{r}) = \mathbf{\nabla}\times(\mathbf{r}\psi_m), \quad \mathbf{N}_{lm}(\mathbf{r}) = k^{-1}\nabla \times \nabla \times (\mathbf{r}\psi_m), \tag{1.56}$$

$$\psi_m(\mathbf{r}) = h_l^{(2)}(kr)Y_{lm}(\mathbf{r}) \tag{1.57}$$

Here we choose a solution with radial dependence determined by the spherical Bessel function of the third kind, $h_l^{(2)}(kr)$ (also known as a Hankel function). See Jackson (1998, Ch. 16.).

There is a bit of a disconnect here

we get

$$\mathbf{\nabla} f = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[Y_{lm}\mathbf{\nabla} f_{lm} + f_{lm}\mathbf{\nabla} Y_{lm}\right] = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[f'_{lm}(r)\,\mathbf{e}_r\,Y_{lm} + f_{lm}(r)\,\mathbf{\nabla} Y_{lm}\right] \tag{1.51}$$

where $f'$ denotes the radial derivative of $f$ and $\mathbf{e}_r$ is a unit vector in the radial direction. This shows that, in addition to the radial $\mathbf{e}_r$, we need two non-radial vectors implicit in the term $\mathbf{\nabla} Y_{lm}$ to make this representation work. See also Phinney and Burridge (1973).

14 Jackson defines a differential operator $\mathbf{L} = i\mathbf{r} \times \nabla$ which corresponds to the basis vector $\mathbf{\Phi}_{lm}$. (The appearance of the pre-multiplying '$i$' should occasion no concern as the Spherical harmonics are themselves complex function having real and imaginary parts).

The component $\mathbf{r}\times\nabla Y_{lm}(\theta,\phi))$ is manifestly orthogonal to the radial direction and so the components of a transverse electric or magnetic field can be written as

$$\text{either}\quad \mathbf{E}_{lm} \propto \mathbf{r}\times\nabla Y_{lm}(\theta,\phi) \tag{1.58}$$
$$\text{or}\quad \mathbf{B}_{lm} \propto \mathbf{r}\times\nabla Y_{lm}(\theta,\phi). \tag{1.59}$$

(Since these two vectors are orthogonal in an electromagnetic plane wave, we clearly cannot have both.) This is fundamental to describing the orientation of a polarisation vector of an electromagnetic wave on the celestial sphere.

### 1.4.2 Representing a general vector field

For completeness we give an example of the representation of a velocity field $\mathbf{u}(r,\theta,\phi)$ in terms of vector spherical harmonics. The general representation of the field $\mathbf{u}$ will be of the form

$$\mathbf{u}(r,\theta,\phi) = \mathbf{e}_r U(r,\theta,\phi) + \boldsymbol{\nabla}' V(r,\theta,\phi) - \mathbf{e}_r \times \boldsymbol{\nabla}' W(r,\theta,\phi) \tag{1.60}$$

for some functions $U, V, W$, where the gradient operator $\boldsymbol{\nabla}'$ acts only in the plane perpendicular to the direction $\mathbf{r}$. Written out explicitly we have

check this equation - there was an ambiguity

**Orthogonal covariant derivatives on the surface of a sphere**

$$\boldsymbol{\nabla}' = \mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}, \qquad \mathbf{e}_r \times \boldsymbol{\nabla}' = -\mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \mathbf{e}_\phi \frac{\partial}{\partial\theta}. \tag{1.61}$$

Here we see the basis vectors appearing in equation (1.52) expressed as the differential operators of (1.53). These are the covariant derivatives in the tangent plane to the sphere. We now get to equation (1.49) by writing down the usual *scalar* spherical expansion (1.50) for each of the the functions $U, V, W$.

We can rewrite (1.60) in terms of the basis functions we have

$$\mathbf{u}(r,\theta,\phi) = \sum_{lm} U_{lm}\mathbf{P}_{lm} + V_{lm}\mathbf{Q}_{lm} + W_{lm}\mathbf{R}_{lm} \tag{1.62}$$

where

$$\mathbf{P}_{lm} = \mathbf{e}_r Y_{lm} \tag{1.63}$$
$$\mathbf{Q}_{lm} = \frac{1}{\sqrt{l(l+1)}}\left[\mathbf{e}_\theta\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] Y_{lm} \tag{1.64}$$
$$\mathbf{R}_{lm} = \frac{1}{\sqrt{l(l+1)}}\left[\mathbf{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \mathbf{e}_\phi\frac{\partial}{\partial\theta}\right] Y_{lm} \tag{1.65}$$

(see equations 1.61).

## 1.5 Representing tensor fields on a sphere

The next step of analysis on a sphere is to go to the second rank tensor harmonics which are used to describe the distribution of fields of arbitrary spin on the sphere. The relevance of this to the CMB is that the polarisation of the CMB polarisation is described in terms of the Stokes parameters $Q, U$ which transform, not as a vector, but as a rank-2 tensor having complex coefficients.

Spherical harmonic analysis of such spin-2 fields were first discussed in the context of the CMB by Seljak and Zaldarriaga (1997) and Zaldarriaga and Seljak (1997, see Appendix A). Here we try to throw a little light on this rather technical area of analysis of functions on a sphere, without going into the details of the derivations.[15]

### 1.5.1 Spin-weighted spherical harmonics

The tensor harmonics are probably best introduced as the rank-$s$ tensor analogues ${}_{(s)}P_l^m(x)$ of the associated Legendre functions $P_l^m(x)$ (equations 1.6 and 1.10). They are the solutions of

**Spin-weighted associated Legendre functions**

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{(m+sx)^2}{1-x^2} + s\right] {}^{(s)}P_l^m(x) = -(l-s)(l+s+1)\,{}^{(s)}P_l^m(x) \quad (1.66)$$

$$l = 0, 1, 2, \ldots, \quad m = 0, \pm1, \pm2, \ldots, \pm l,$$

$$s = 0, \pm1, \pm2, \ldots, \quad |s| \le l$$

$$\text{with boundary condition:} \quad {}^{(s)}P_l^m(x) \text{ finite at } x = \pm 1.$$

The solutions can be generalised to include half-integer $l, m, s$ (the *spinor harmonics*).

The parameter $s$ is called the *spin parameter* since it is related to spin states in quantum mechanics. This equation corresponds to the polar $x = \cos\theta$ part of the solution of the Helmholtz equation $(\nabla^2 + \lambda^2)\,{}^{(s)}\mathbf{H} = 0$ for a general rank-$s$ tensor field ${}^{(s)}\mathbf{H}$ on a sphere (the values of $\lambda$ are fixed by the boundary conditions in the usual manner).[16]

[15] Much of the early work on tensor spherical harmonics came as a by-product of work on angular momentum and representations of the rotation group in quantum mechanics. That early work culminated in the book by Edmonds (1996, first published 1957) and that led to the paper on the general tensor spherical harmonic functions of Goldberg et al. (1967). This is the source most often cited in the CMB related literature. However, there was another line of development in the world of Geophysics, starting with mathematical papers by Backus (1966, 1967). This was taken up in a more practical sense by Burridge (1969) and by Phinney and Burridge (1973), and then by James (1976) who introduced a slightly different but important recursive definition of the tensor spherical harmonics. Here, I follow the geophysics line of development.
There is a thorough coverage of spin-weighted spherical and spheroidal harmonics in the book by Breuer (1975, §6.11, p.104 *et seq.* and Appendix B. ), though the typewriter font used for text and equations makes it difficult to read. There is a substantial collection of results on these functions in Winch and Roberts (1995).

[16] Here I have chosen to put the spin parameter $s$ in parentheses and use it as a raised index appearing before the symbol, as in ${}^{(s)}P_l^m$ and ${}^{(s)}Y_{lm}$. This emphasises the fact that, while the ${}^{(s)}P_l^m$ are mutually orthogonal with

The corresponding spin-weighted spherical harmonics, which are functions of $(\theta, \phi)$, are then

**Spin-weighted spherical harmonics**

$$^{(s)}Y_{lm}(\theta, \phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} {}^{(s)}P_l^m(\cos\theta)\, e^{im\phi} \tag{1.67}$$

The spin-1 and spin-2 harmonics are

***spin*-1**: $s = 1, l = 1$

$$^1Y_{1,0} = \sqrt{\frac{3}{8\pi}}\sin\theta$$

$$^1Y_{1,\pm1} = -\sqrt{\frac{3}{16\pi}}(1 \mp \cos\theta)\, e^{\pm i\phi}$$

***spin*-2**: $s = 2, l = 2$

$$^{(2)}Y_{2,0} = \sqrt{\frac{45}{96\pi}}\sin^2\theta$$

$$^{(2)}Y_{2,\pm1} = \sqrt{\frac{5}{16\pi}}\sin\theta(1 \mp \cos\theta)\, e^{\pm i\phi}$$

$$^{(2)}Y_{2,\pm2} = \sqrt{\frac{5}{64\pi}}(1 \mp \cos\theta)^2\, e^{\pm 2i\phi} \tag{1.68}$$

where we have written $^sY_{l,m}$ for $^sY_{lm}$ and likewise $Y_{l,m}$ for $Y_{lm}$. The expressions for negative spin are found by use of the parity and conjugacy relationships given below (equations 1.70, 1.71), while for zero spin these are the regular spherical harmonics as given in equations (1.24).

## 1.5.2 Some properties of the $^{(s)}Y_{lm}$

The functions $^{(s)}Y_l^m(\theta, \phi)$ satisfy the orthogonality relationships:

$$\int_{S_2} {}^{(s)}Y^*_{lm}(\theta, \phi)\, {}^{(s)}Y_l^m(\theta, \phi)\, \sin\theta d\theta\, d\phi = \delta_{l'l}\delta_{m'm} \tag{1.69}$$

where $^{(s)}Y^*_{lm}$ is the complex conjugate of $^{(s)}Y_l^m$. For a given value of $s$ the spin-weighted harmonics form a complete set of orthogonal functions on the sphere. Note that orthogonality does not extend to the spin parameter $s$.

They also satisfy the parity relationship

$$^{(s)}Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l\, {}^{(-s)}Y_{lm}(\theta, \phi) \tag{1.70}$$

and the conjugacy relationships

$$^{(s)}Y^*_{lm} = (-1)^{m+s}\, {}^{(-s)}Y_{l\,-m} \tag{1.71}$$

respect to the indices $l, m$, they are orthogonal with respect to the spin parameter $s$. The more usual notation is to write these as $P_{s,l}^{\,m}$ and $_sY_{lm}$ or $Y_{s,lm}$.

The analogue of equation (1.21) for the scalar spherical harmonics[17] is

$$ {}^{(s)}Y_{lm}(\theta,\phi) = \left[\frac{(l+m)!(l-m)!}{(l+2)!(l-s)!}\frac{2l+1}{4\pi}\right]^{1/2} \sin^{2l}\left(\frac{\theta}{2}\right) \times \sum_r \binom{l-s}{r}\binom{l+2}{r+s-m}(-1)^{l-r-s}e^{im\phi}\cot^{2r+s-m}\left(\frac{\theta}{2}\right) \qquad (1.72)$$

This is a somewhat unprepossessing equation and so one of the main goals will be to do the analysis on a sphere using the ${}^{(s)}Y_{lm}(\theta,\phi)$, as we must since we are analysing tensor fields, and then, for practical purposes, express these functions in terms of the standard, spin-0 spherical harmonics, $Y_{lm}(\theta,\phi)$. [18]

This discussion parallels that of the vector spherical harmonics in representing a vector field on the sphere through equations (1.60 - 1.65).

### 1.5.3 Relating ${}^{(s)}Y_{lm}$ to $Y_{lm}$

We might ask what are the equivalents of equation (1.19, 1.63, 1.64, 1.65 ) for spin-$s$ spherical harmonics. The best way of writing this down is to define two new differential operators ð and $\bar{ð}$ which are the generalisations of the operators (1.64) and (1.65) that occur in the spin-0, *i.e.* standard, $Y_{lm}$'s:

$$ ðf = -(\sin\theta)^s \left\{\frac{\partial}{\partial\theta} + i\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}\left[(\sin\theta)^{-s}f\right] \qquad (1.73)$$

$$ \bar{ð}f = -(\sin\theta)^{-s} \left\{\frac{\partial}{\partial\theta} - i\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}\left[(\sin\theta)^{s}f\right] \qquad (1.74)$$

(Goldberg et al., 1967, equations (2.3),(2.3a)). This form shows explicitly the reason that these are referred to as being *spin-weighted*. On the other hand, these definitions are equivalent to the somewhat simpler and more familiar looking

$$ ðf = -\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - s\cot\theta\right]f \qquad (1.75)$$

$$ \bar{ð}f = -\left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + s\cot\theta\right]f \qquad (1.76)$$

These differential operators are, in effect, the covariant derivatives on the surface of the sphere that are relevant to spin-$s$ functions,[19] just as equations (1.64) and (1.65) are the covariant derivatives of scalar functions on the surface of a sphere. The differential operators ð, $\bar{ð}$ are linear and satisfy the Leibniz rule.

[17] This is the version corresponding to the Condon-Shortley phase choice.

[18] Since the basic equation (1.66) is linear, any linear combination of the ${}^{(s)}Y_{lm}$ solutions is also a solution. James (1976) introduced particular linear combinations, denoted by $Y_{n(k)}$, where $n(k)$ was short for the string of indices $i, i+1, \ldots, i+k$, that could be generated iteratively: $Y_{n(k)}$ could be expressed in terms of the $Y_{n(k-1)}$. These linear combinations are also discussed by Newman and Silva-Ortigoza (2006) in the context of gravitational radiation. This variety of spherical harmonics ${}^{(s)}Y_{lm}$ is referred to as *tensor spherical harmonics*, as opposed to the ${}^{(s)}Y_{lm}$ which are the *spin-weighted spherical harmonics*.

[19] In principle, since the form of the operator ð depends explicitly on $s$, we might decorate the symbol ð with a subscript or superscript to indicate the $s$-dependence, *e.g.* ${}^{(s)}ð$. By convention this is not done since it would render the notation somewhat cumbersome.

The ${}^{(s)}Y_{lm}$ are related to the familiar scalar field $Y_{lm}$'s by

$$ {}^{(s)}Y_{lm} = \sqrt{\frac{(l-s)!}{(l+s)!}}\,(\eth)^s\, Y_{lm} \qquad 0 \le s \le l, \tag{1.77}$$

$$ {}^{(s)}Y_{lm} = \sqrt{\frac{(l+s)!}{(l-s)!}}\,(-\bar{\eth})^{-s}\, Y_{lm} \quad -l \le s \le 0, \tag{1.78}$$

$$ {}^{(s)}Y_{lm} = 0, \qquad\qquad |\,s\,| > l. \tag{1.79}$$

In relating the spin spherical harmonics to the familiar $Y_{lm}$ and their derivatives, we are in a position to describe vector and tensor fields on the sphere in terms of the $Y_{lm}$ and their derivatives. This is the tensor field generalisation of equations (1.61) which describe the representation of a vector field on the sphere in terms of the $Y_{lm}$ and their derivatives.

As a consequence of these last equations we have the important identities:

$$\eth\,({}^{(s)}Y_{lm}) = +\sqrt{(l-s)(l+s-1)}\;{}^{(s+1)}Y_{lm} \tag{1.80}$$

$$\bar{\eth}\,({}^{(s)}Y_{lm}) = -\sqrt{(l+s)(l-s+1)}\;{}^{(s-1)}Y_{lm} \tag{1.81}$$

In the language of quantum mechanics, we see that $\eth({}^{(s)}Y_{lm})$ raises the spin quantum number $s$ and $\bar{\eth}({}^{(s)}Y_{lm})$ lowers it. It can further be shown that

$$\bar{\eth}\eth\;{}^{(s)}Y_{lm} = -(l-s)(l+s+1)\;{}^{(s)}Y_{lm} \tag{1.82}$$

and so the spin-$s$ harmonic functions ${}^{(s)}Y_{lm}$ are eigenfunctions of the operator $\bar{\eth}\eth$. In fact, if we write out this last equation in full detail we have

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{(m+s\cos\theta)^2}{\sin^2\theta}\right]{}^{(s)}P_l^m = -(l-s)(l+s+1){}^{(s)}P_l^m \tag{1.83}$$

which, with $x=\cos\theta$, is just our starting point (1.66) defining the spin-weighted Legendre functions. The relationship between the ${}^{(s)}P_l^m(\cos\theta)$ and the ${}^{(s)}Y_{lm}(\theta,\phi)$ is given in equation (1.67).

In addition to equation (1.82) we have

$$\eth\,\bar{\eth}\;{}^{(s)}Y_{lm} = -(l+s)(l-s+1)\;{}^{(s)}Y_{lm}, \tag{1.84}$$

which leads us to the commutation relationships for these spin operators:

$$\left[\bar{\eth}\eth - \eth\,\bar{\eth}\right]\;{}^{(s)}Y_{lm} = 2s\,{}^{(s)}Y_{lm} \tag{1.85}$$

Since the ${}^{(s)}Y_{lm}$ are a complete set of orthogonal functions of spin $s$, any spin-$s$ function will satisfy this relationship. Finally there is the result that

$$\overline{\eth\eth}\eth\eth Y_{lm} = \eth\eth\overline{\eth\eth} Y_{lm} = (l+2)(l+1)l(l-1)Y_{lm} \tag{1.86}$$

This equation is used when it comes to separating out the $E$-modes and $B$-modes of the polarisation of the CMB (Bunn et al., 2003).

## 1.6 Harmonic analysis of the CMB $T, Q, U$ maps

We have three parameters describing the CMB in directions $\mathbf{n}$ on the sky: the temperature $T(\mathbf{n})$, and the two Stokes parameters $Q(\mathbf{n}), U(\mathbf{n})$. The last two are combined into the complex quantities $(Q \pm iU)(\mathbf{n})$, which transform as spin $\pm 2$ functions.[20] So their representation in terms of spherical harmonics is

$$T(\mathbf{n}) = \sum_{lm} a_{T,lm}\; Y_{lm}(\mathbf{n}) \tag{1.87}$$

$$(Q + iU)(\mathbf{n}) = \sum_{lm} {}^{(+2)}a_{lm}\; {}^{(+2)}Y_{lm}(\mathbf{n}) \tag{1.88}$$

$$(Q - iU)(\mathbf{n}) = \sum_{lm} {}^{(-2)}a_{lm}\; {}^{(-2)}Y_{lm}(\mathbf{n}) \tag{1.89}$$

The inverse of equations (1.87, 1.88, 1.89) gives the spherical harmonic representation of the CMB signals $T, Q, U$:

$$a_{T,lm} = \int T(\mathbf{n}) Y^*_{lm}(\mathbf{n})\, d\Omega, \tag{1.90}$$

$${}^{(+2)}a_{lm} = \int (Q + iU)\; {}^{(+2)}Y^*_{lm}(\mathbf{n})\, d\Omega, \qquad {}^{(-2)}a_{lm} = \int (Q - iU)\; {}^{(-2)}Y^*_{lm}(\mathbf{n})\, d\Omega. \tag{1.91}$$

The last two equations express the spherical harmonic transform of the $Q, U$ maps in terms of the spin-weighted spherical harmonics. It is, however, useful to express the ${}^{(\pm 2)}Y^*_{lm}(\mathbf{n})$ functions in terms of the more familiar and easier to handle $Y_{lm}(\mathbf{n})$. Once we have done that we shall split the polarisation map into its 'electric' and 'magnetic' components. In the ideal situation that the sky coverage is complete, and the data noise-free, these components are mutually independent.

### 1.6.1 Spherical harmonic transform of the $Q \pm iU$ map

These relationships allow us to express the polarisation parameters $(Q, U)$ as given in (1.88) and (1.89) directly in terms of the more familiar spin-0 $Y_{lm}$'s by using equations (1.77, 1.78) with $s = 2$:[21]

$$\bar{\eth}^2 (Q + iU)(\mathbf{n}) = \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}}\; {}^{(2)}a_{lm} Y_{lm}(\mathbf{n}) \tag{1.93}$$

$$\eth^2 (Q - iU)(\mathbf{n}) = \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}}\; {}^{(-2)}a_{lm} Y_{lm}(\mathbf{n}) \tag{1.94}$$

[20] Here we follow closely the development by Zaldarriaga and Seljak (1997, §II).

[21] In deriving these we need to note that the square-root term comes from

$$(l+2)!/(l-2)! = \sqrt{(l+2)(l+1)l(l-1)} \tag{1.92}$$

which arises when doing the derivative $\bar{\eth}$ twice.

We can then invert these equations in the usual way by multiplying both sides with $Y^*_{lm}$ and integrating over the sphere. We get

$$^{(2)}a_{lm} = \sqrt{\frac{(l-2)!}{(l+2)!}} \int Y^*_{lm}(\mathbf{n})\, \bar{\eth}^2 (Q + iU)(\mathbf{n})\, d\Omega \tag{1.95}$$

$$^{(-2)}a_{lm} = \sqrt{\frac{(l-2)!}{(l+2)!}} \int Y^*_{lm}(\mathbf{n})\, \eth^2 (Q - iU)(\mathbf{n})\, d\Omega \tag{1.96}$$

This gives us what we want: the harmonic components of the polarisation field expressed in terms of the data. In practice, however, the need to differentiate the data, as in the term $\bar{\eth}^2(Q+iU)(\mathbf{n})$, brings with it a number of practical problems. The solution is to deal directly with the spin-weighted $^{(2)}a_{lm}Y_{lm}$.

### 1.6.2 'Electric' and 'Magnetic' components of polarisation

The weight-2 coefficients $^{(\pm 2)}a_{lm}$ can be combined to produce the even and odd parity coefficients[22]

$$a_{E,lm} = -[^{(+2)}a_{lm} + {}^{(-2)}a_{lm}]/2 \tag{1.97}$$

$$a_{B,lm} = i\ [^{(+2)}a_{lm} - {}^{(-2)}a_{lm}]/2 \tag{1.98}$$

These describe the distribution of the even, 'electric', and odd, 'magnetic', parity components of the polarisation field,[23] Under a parity transformation the $E$-mode component of the polarisation field does not change sign, while the $B$-mode component does change sign. This is precisely the same behaviour as the $\mathbf{E}, \mathbf{B}$ components of the electromagnetic field and this is what gives rise to referring to these as the *E-mode and B-mode components of the CMB polarisation*.

## 1.7 The angular power spectra

We now turn to applying the theory of spherical harmonics to the CMB sky.

The basic statistical descriptor of any zero-mean random field on the sky is its power spectrum, $C_l$, which describes the contribution of the means square amplitude of spherical

[22] The coefficients $^{(\pm 2)}a_{lm}$ have the symmetries

$$a^*_{T,lm} = a_{T,l\,-m}, \quad {}^{(-2)}a^*_{lm} = {}^{(+2)}a_{l\,-m},$$

where the superscript $^*$ denotes complex conjugation.

[23] The *parity transformation*, $P$, involves changing the sign of one of the space coordinates: $P : (x, y, z) \to (-x, y, z)$. In effect this is 'physics in a mirror world': addressing the question of what happens to the laws of physics when a coordinate is flipped. Newton's Laws of motion are the same in the mirror world, they are invariant under the parity transformation. Some vectors flip under parity transformation, others do not. The velocity of a particle changes sign under $P$, while the angular momentum vector does not change sign under $P$ (a rotating wheel rotates in the same direction in the mirror world). Vectors that do not flip under parity transform, like the angular momentum and magnetic field vectors, are called *axial vectors* or *pseudo-vectors*.

harmonic modes of frequency $l$ to the total variance. For a given value of $l$ there are $2l + 1$ contributors corresponding to the range of the index $m$ in defining the $a_{lm}$.

### 1.7.1 Angular Power Spectrum

The way these are used is as follows. Suppose we wish to represent a function $f(\theta, \phi)$ defined on a sphere by its spherical harmonic expansion:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \; Y_{l,m}(\theta, \phi) \tag{1.99}$$

then the coefficients $a_{lm}$ are given by

$$a_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; f(\theta, \phi) \; Y^*_{l,m}(\theta, \phi) \tag{1.100}$$

The *angular power spectrum* of the function $f(\theta, \phi)$ is defined as a function of the multipole index, $l$:

$$c_l = \sum_{m=-l}^{l} a_{lm}^2 \tag{1.101}$$

This is just the sum of the squares of the $(2l+1)$ coefficients $a_{lm}$ belonging to a given value of $l$. In this way $l$ plays a role analogous to the frequency in Fourier analysis.

The total power of the function $f(\theta, \phi)$ is then defined as

$$\sum_{l=0}^{\infty} c_l = \frac{1}{4\pi} \int_{\Omega} f(\Omega)^2 \, d\Omega \tag{1.102}$$

where the last equality follows from Parseval's theorem.

If $f(\theta, \phi)$ is a realisation of a random process. then different realisations have different $f(\theta, \phi)$ and yield different $c_l$. The coefficients $c_l$ of equation (1.101) are specific to a particular realisation. The statistical average of the $c_l$ over many realisations, if it converges, is the expected power spectrum and is denoted by $C_l$. Parametrised theoretical models for $f(\theta, \phi)$ provide parametrised models for the expected power spectrum. Determining the model parameters from an observation of a particular realisation $f(\theta, \phi)$ is a problem in statistical inference.

### 1.7.2 The Power Spectrum of the sky map

Now we turn to analysing the distribution on the sky of the fluctuations, $\Delta T(\theta, \phi)/T$ in the cosmic background radiation temperature, $T(\theta, \phi)$. It is conventional to normalise these fluctuations relative to the mean all sky temperature, $T_0$:

$$\frac{\Delta T(\theta, \phi)}{T} = \frac{T(\theta, \phi) - T_0}{T_0} \tag{1.103}$$

where on the left side we drop the subscript on $T_0$ for convenience.[24] $\Delta T/T$ is a random function of position on the sky. It has, by construction, zero mean. Given the data we can calculate its spherical harmonic representation:

$$a_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \frac{\Delta T(\theta,\phi)}{T} \; Y^*_{l,m}(\theta,\phi) \tag{1.104}$$

The $a_{lm}$ are independently distributed complex-valued random variables having zero mean and variance

$$C_l = \langle |a^2_{lm}| \rangle \tag{1.105}$$

where $\langle X \rangle$ denotes the statistical expectation of the random variable $X$. Note that if the temperature fluctuations are $\Delta T$ are Gaussian distributed, then so are the $a_{lm}$, and hence the $C_l$ are chi-square distributed with $2l+1$ degrees of freedom.[25]

In practice we have to estimate $C_l$ from the given data. This can be done by averaging the measured $a_{lm}$ for each $l$ over its $2l+1$ over $m$-values:

$$\bar{C_l} = \frac{1}{2l+1} \sum_{m=-l}^{l} |a^2_{lm}| \tag{1.106}$$

This is an unbiased estimator. We see from these last two equations that the contribution to the variance from mode $l$ is then

$$\langle |a^2_{lm}| \rangle = (2l+1)\bar{C_l} \tag{1.107}$$

We understand this because there are $2l+1$ modes that contribute to the variance at each $l$.

In the cosmological context we are interested in related the observed power spectrum of the temperature fluctuations to the 3-dimensional power spectrum of the underlying temperature field and relating that to the power spectrum of the density fluctuations. This is discussed in section 1.9.

### 1.7.3 Correlation Function of the sky map

The use of the angular correlation function to determine the power spectrum, $C_l$, of the CMB temperature fluctuations was introduced by (Szapudi et al., 2001b,a) and later generalised to handling polarisation data by Chon et al. (2004). For simplicity we shall deal only with the temperature fluctuations.

The correlation between of the temperature variations $\Delta T(\theta,\phi)/T$ in directions on the sky separated by some angle $\psi$ is defined as the average[26]

$$\xi(\psi) = \left\langle \frac{\Delta T(\mathbf{n}_1)}{T} \frac{\Delta T(\mathbf{n}_2)}{T} \right\rangle, \quad \mathbf{n_1}\cdot\mathbf{n_2} = \cos\psi \tag{1.108}$$

where the average is taken over all pairs of directions $\mathbf{n}_1, \mathbf{n}_2$ separated by an angle $\psi$.

[24] We often see the notation $\Theta = \Delta T/T$ in the literature.

[25] See also Peebles (1993, Ch. 21, equations 21.78 *et seq.*) where $C_l$ here is denoted by $a_l^2$.

[26] The correlation function of the relative temperature fluctuations is frequently denoted by $C(\psi)$. Here we use $\xi(\psi)$ in order to clearly distinguish from the symbol $C_l$ which denotes the amplitude of the power spectrum at a given value of $l$.

The spherical coordinate analogue of the well-known statement that the correlation function is the Fourier transform of the power spectrum then reads [27]

$$\xi(\psi) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l P_l(\cos\psi) \tag{1.110}$$

$$C_l = 2\pi\int_0^{\pi}\xi(\psi)\, P_l(\cos\psi)\, \sin\psi d\psi \tag{1.111}$$

The fact that the correlation function $\xi(\psi)$ depends only on the angle $\psi$ between the two directions $\mathbf{n}_1, \mathbf{n}_2$ is a reflection of the assumed statistical isotropy of the underlying random process $\Delta T(\mathbf{n})/T$.

The variance of the fluctuations is just $\xi(0)$ and is given by

$$\xi(0) = \frac{1}{4\pi}\sum_l (2l+1)C_l \tag{1.112}$$

If we approximate this expression as an integral we have

$$\xi(0) = \frac{1}{4\pi}\sum_l (2l+1)C_l \simeq \frac{1}{2\pi}\int l(l+1)C_l \, d\ln l \tag{1.113}$$

This shows that $l(l+1)C_l/2\pi$ is the contribution of the mode $l$ to the total variance of the field, per unit logarithmic interval $d\ln l$. Accordingly it is usual to plot the values of $l(l+1)C_l/2\pi$ versus $\log l$ when presenting power spectrum data.

## 1.8 Apodisation

In practice, we only know the power spectrum for a finite range of $l$-values, and we can only we measure the correlation function over a finite range of angles angles $\psi$: the practical ranges of $l$ and $\psi$ have cut-offs.

If there is no data at large angular scales, say $\psi > \psi_{max}$, we have to truncate the integral in equation (1.111). But a sharp cut-off, *i.e.* simply putting the upper limit of integration to $\psi_{max}$, would generate unphysical artefacts in $\xi(\psi)$. Hard cut-offs cause unphysical high frequency oscillations in the transformed function. To avoid this we need to introduce an *apodization function*.[28]

[27] We derive this as follows:

$$\xi(\psi) = \langle \Delta T(\mathbf{n}_1)\Delta T(\mathbf{n}_2)\rangle = \sum_{lm}\sum_{l'm'} \underbrace{\langle a_{lm} a^*_{l'm'}\rangle}_{C_l\delta_{ll'}\delta_{mm'}} Y_{lm}(\mathbf{n}_1)Y^*_{l'm'}(\mathbf{n}_2)$$

$$= \sum_l C_l \underbrace{\sum_m Y_{lm}(\mathbf{n}_1)Y^*_{lm}(\mathbf{n}_2)}_{\frac{1}{4\pi}(2l+1)P(\cos\psi)} = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l P_l(\cos\psi). \tag{1.109}$$

[28] Also known simply as a *window function*.

If we have data for values of $\psi$ up to $\psi_{max}$ we calculate the $C_l$ using the equation

$$C_l^{apodized} = 2\pi \int_{\psi=0}^{\psi_{max}} \xi(\psi)\, F(\psi)\, P_l(\cos\psi)\ \sin\psi d\psi \tag{1.114}$$

where $F(\psi)$ denotes the apodization function. $F(\psi)$ should have the property that $F(0) = 1$ and goes to zero smoothly around $\psi_{max}$. There is a considerable literature on this subject providing a large number of alternative functions that are suited to different applications. A typical $F(\psi)$ is the 'cosine bell':

$$F(\psi) = \frac{1}{2}\left[1 + \cos\left(\pi \frac{\psi}{\psi_{apod}}\right)\right] \tag{1.115}$$

where $\psi_{apod}$ is close to $\psi_{max}$.

Notice that apodization has an effect on the contribution of $\xi(\psi)$ to each of the coefficients $C_l$ (see equation 1.114). The cost of the apodization is that the coefficients $C_l^{apodized}$ are not statistically independent: they have a covariance matrix with non-zero off-diagonal terms.

### 1.8.1 Covariance of apodized $C_l$'s

The covariance between the $C_l$'s for $l$-values $l$ and $l'$ can be denoted by $C_{ll}$, and can be computed from a large number of realisations of simulated data. If we label the values of the $c_l$ derived from the $i^{th}$ realisation by $C_l^{(i)}$ we have

$$C_{ll'} = \frac{1}{N}\sum_{i=1}^{N}(C_l^{(i)} - \bar{C}_l)(C_{l'}^{(i)} - \bar{C}_{l'}), \quad \bar{C}_l = \frac{1}{N}\sum_{i=1}^{N} C_l^{(i)} \tag{1.116}$$

If this is further normalised relative to the variances of the $C_l$ and $C_{l'}$, we have the statistical correlation function of the $C_l$ and $C_{l'}$.

The covariance between the $C_l$ and $C_{l'}$ is of importance when considering data arising from selected areas or from maps from which certain regions are to be excluded (like the Galactic plane in CB maps). Under these circumstances, the $C_l$ are not independent, and the covariance function provides a measure of the degree of inter-dependence of the modes.

## 1.9 The power spectrum

Suppose that the function $f(\theta, \phi)$ of equation (1.99) is a 2-dimensional sample of the function values on a spherical surface drawn in a 3-dimensional statistically isotropic, zero-mean, random field. Suppose further that the underlying 3-dimensional field has power spectrum $\mathcal{P}(k)$. The question is how the $C_l$ for the 2-dimensional field $f(\theta, \phi)$ relate to the underlying $\mathcal{P}(k)$. The assumption of statistical isotropy allows this relationship to be established.

### 1.9.1 From 3-dimensions to 2-dimensions on a sphere

The key to this is equation (1.40) which tells us how to relate the 3-dimensional spatial Fourier amplitudes of a field to the spherical harmonic representation of the field on the surface of a sphere.

Consider a random function $f(\mathbf{x})$ having Fourier transform $g(\mathbf{k})$ and power spectrum $\mathcal{P}_f(k)$ defined by $\langle g(\mathbf{k})\, g(\mathbf{k}')\rangle = \mathcal{P}_g(k)\, \delta_D(\mathbf{k}-\mathbf{k}')$ ($\delta_D(\mathbf{x})$ here being the Dirac delta function). Collecting up earlier results, $f(\mathbf{x})$ has Fourier and spherical harmonic representations

$$f(\mathbf{x}) = \int g(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}, \qquad f(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\hat{\mathbf{x}}) \tag{1.117}$$

where $\hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. The coefficients $a_{lm}$ are related to the $g(\mathbf{k})$ via the relationship (1.40):

$$a_{lm} = 4\pi\, i^l \int j_l(kr)\, g(\mathbf{k})\, Y^*_{lm}(\hat{\mathbf{k}}) d^3\mathbf{k} \tag{1.118}$$

From this we can write the variance of the $a_{lm}$ as

$$\langle a_{lm} a^*_{lm}\rangle = 16\pi^2 \int\int j_l(kr) j_l(k'r) \langle g(\mathbf{k}) g^*(\mathbf{k}')\rangle\; Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}') d^3\mathbf{k} d^3\mathbf{k}' \tag{1.119}$$

$$= 16\pi^2 \int j_l(kr)^2 \mathcal{P}_g(k)\, Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) k^2 dk\, d\Omega_{\hat{\mathbf{k}}} \tag{1.120}$$

The integral over the directions $\hat{\mathbf{k}}$ simply reduces to unity since the $Y_{lm}$ are orthogonal functions and so we are left with

$$C_l = \langle a_{lm} a^*_{lm}\rangle = 16\pi^2 \int_0^{\infty} j_l(kr)^2\, \mathcal{P}_g(k)\, k^2 dk \tag{1.121}$$

using (1.105) for the definition of the amplitudes $C_l$. The variable $r$ is set to the radius of the sphere on which the $C_l$ are defined.

### 1.9.2 Special simple cases

There is a simple and relevant case that can be treated analytically arising when the 3-dimensional field has a power law power spectrum:

$$\mathcal{P}(k) = Ak^n \tag{1.122}$$

When the field is a field of density fluctuations, $n = 1$ corresponds to the Harrison-Zeldovich initial spectrum. Going through the calculation it is found that

$$C_l = |a_l|^2 = \frac{4A}{2^{3-n}} \frac{\Gamma(3-n)}{\Gamma^2(2-\frac{1}{2}n)} \frac{\Gamma(l+\frac{1}{2}(n-1))}{\Gamma(l+\frac{1}{2}(5-n))} \tag{1.123}$$

If we normalise with respect to $C_2$, we have

$$C_l = C_2 \frac{\Gamma(\frac{1}{2}(9-n))}{\Gamma(\frac{1}{2}(3+n))} \frac{\Gamma(l+\frac{1}{2}(n-1))}{\Gamma(l+\frac{1}{2}(5-n))} \tag{1.124}$$

Equation (1.123) is somewhat intimidating: the Γ-functions are not very intuitive! However, there are some useful and important limiting cases.[29]

For $n = 0, 1$:

$$n = 0: \qquad C_l = 8A\frac{1}{(4l^2-1)(2l+3)}$$
$$n = 1: \qquad C_l = \frac{4A}{\pi}\frac{1}{l(l+1)} \tag{1.125}$$

and for large $l$:

$$C_l \propto l^{n-3}, \quad l \gg 1. \tag{1.126}$$

As expected, the shape of the power spectrum $C_l$ depends on the spectrum of fluctuations. In the large $l$ limit for $n = 0, 1$ we have

$$l^3 C_l \to \text{constant}, \quad l \to \infty, \quad n = 0 \tag{1.127}$$
$$l(l+1)C_l = \text{constant}, \quad n = 1 \tag{1.128}$$

We note from equation (1.113) that $l(l+1)C_l$ is proportional to the contribution of mode $l$ to the total variance per unit logarithmic interval $d\ln l$ in $l$. Hence the 'flat' $n = 1$ spectrum contributes equal power per unit logarithmic interval, $d\ln l$, at all $l$-values.

Translating from $l$ to angular scales $\theta$ using equation (**??**) we then have that at small angular scales

$$\frac{\Delta T}{T} \propto \theta^{\frac{1}{2}(1-n)} \tag{1.129}$$

Following common practice, we have used the symbol $\Delta T/T$ to denote the *r.m.s.* value of the temperature fluctuations.[30]

We have not folded in an instrumental response, so this represents the measurements from perfect equipment. The important thing is that for the $n = 1$, Harrison Zel'dovich Spectrum, the amplitude of the temperature fluctuations is scale independent. Conversely, from observations at multiple angular scales we can hope to determine $n$ from direct observation, or indeed test the proposed form of the primordial power spectrum.

### 1.9.3 The Bispectrum of the distribution on a sphere

The power spectrum coefficients, $C_l$ the expected mean square of the spherical harmonic coefficients: $\langle | a_{lm} a_{lm} | \rangle$. The mean of the product of three coefficients is called the *Bispectrum, $B_{l_1 l_2 l_3}$*, and is defined by the equation

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = B_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{1.130}$$

[29] Two derive these limiting case we need the identity $\Gamma(z+1) = z\Gamma(z)$ (Abramowitz and Stegun, 1965, eq. 6.1.15) and Stirling's large $z$ approximation $\Gamma(z) \sim (2\pi)^{\frac{1}{2}} e^{-z} z^{z-\frac{1}{2}}(1 + O(z^{-1}))$ (Abramowitz and Stegun, 1965, eq. 6.1.37) to zeroth order.

[30] We have the following heuristic argument. If the temperature fluctuations on scale $\lambda$ are associated with fluctuations in the gravitational potential $\Delta\phi_\lambda$ on that scale, then $\Delta T/T \sim \Delta\phi_\lambda/c^2$. We can estimate $\Delta\phi_\lambda \sim G\delta M/\lambda \sim G\delta\rho_\lambda\lambda^2$ where $\delta\rho_\lambda$ is the typical density fluctuation on scale $\lambda$. If the power spectrum is $\mathcal{P}(k) \sim k^n$ then $\delta\rho_\lambda \sim \lambda^{-(3+n)/2}$ and hence $\Delta T/T \sim \lambda^{-(1-n)/2} \sim \theta^{(1-n)/2}$, where $\theta$ is the angle subtended by the scale $\lambda$.

where the last term is the Wigner $3j$-symbol (see equation 1.31).

## 1.10 Fitting the CMB power spectrum

This section could be moved to analysis section

The key datum extracted from the CMB maps is the power spectrum of the temperature fluctuations. If the fluctuations are statistically isotropic and Gaussian distributed, the power spectrum $C_l$ is sufficient to completely characterise the statistical properties of the map. The cosmological models that are to be tested will be parametrised by a set of cosmological parameters which we can denote by $\{\theta_i\}$, and for each model we can compute the probability $\mathbb{P}[C_l \,|\, \theta_i\,]$ that the measured $C_l$ come from that model. By reducing the number of bins the computational effort in the search for the best fit is considerably reduced.

### 1.10.1 CMB data compression

The power spectrum data from CMB experiments is presented somewhat differently from the idealised discussion of the previous sections. Firstly, what is plotted is the 'flattened' spectrum

$$D_l = \frac{1}{2\pi} l(l+1) C_l \tag{1.131}$$

Following (1.106), the power spectrum for the $D_l$ is then defined as

$$\bar{D}_l = \frac{1}{2\pi} \frac{l(l+1)}{2l+1} \sum_m |\, a_{lm}\,|^2 \tag{1.132}$$

(Bond et al., 2000, their notation is a little different from that used here: they use $C_l$ for $D_l$ and $\hat{C}_l$ for $\bar{D}_l$.). The cosmological parameters will be derived from fitting models to the $\bar{D}_l$ rather than the $\bar{C}_l$.

A further simplification is achieved by binning the $\bar{D}_l$ values into suitably chosen ranges of $l$-values (Bartlett et al., 2000). Most of the graphs showing the CMB power spectrum are plots of $D_l$ that have been binned into blocks of $l$-values. The simplification arises because the cosmological models then only generate theoretical values for the binned $D_l$, making the fitting procedure more efficient. The binning process reduces some $\sim 1000$ $l$-values to fewer than $50 - 100$ binned values, and is referred to a 'data compression'.

### 1.10.2 CMB likelihood

The way the best fitting model parameters $\{\theta_i\}$ will be determined is based on maximising the likelihood that the data, *i.e.* a temperature fluctuation map or the observed $C_l$, arises from a particular set of cosmological parameters.

If we describe the data by a vector $\mathbf{d}$ and assume that the underlying statistical distribution of the measurements $\mathbf{d}$ is Gaussian with covariance $\mathbf{C}$, then

$$\mathbb{P}[\,\mathbf{d}\,|\,\text{model}\,] = \frac{1}{\sqrt{(2\pi)^M \,|\,\mathbf{C}\,|}} \exp\left[-\frac{1}{2}\mathbf{d}^T\mathbf{C}^{-1}\mathbf{d}\right] \tag{1.133}$$

Here $M$ is the dimension of **d**, which is the number of pixels in the map when **d** describes the temperature map. There are two contributions to the data: there is the realisation **s** of the CMB temperature map that we are measuring, and there is noise **n** arising from non-cosmological interference: $\mathbf{d} = \mathbf{s} + \mathbf{n}$. The covariance matrix **C** is then

$$\mathbf{C} = \langle \mathbf{d}\mathbf{d}^T \rangle = \langle \mathbf{s}\mathbf{s}^T \rangle + \langle \mathbf{n}\mathbf{n}^T \rangle \tag{1.134}$$

$$= \mathbf{S} + \mathbf{N} \tag{1.135}$$

The first line follows because **s** and **n** are independent. Using the result from matrix theory:

$$|\, e^{\mathbf{X}} \,| = e^{\mathrm{tr}\,\mathbf{X}} \tag{1.136}$$

Something wrong with layout! Fiddled $\mathbb{P}[\dots]$

we can rewrite (1.133) as

$$\mathbb{P}[\mathbf{d} \,|\, \text{model}\,] = \frac{1}{\sqrt{(2\pi)^N}} \exp -\frac{1}{2}\left[\mathbf{d}^T(\mathbf{S}+\mathbf{N})^{-1}\mathbf{d} + \mathrm{tr}\,\mathbf{ln}(\mathbf{S}+\mathbf{N})\right] \tag{1.137}$$

We see in equation (1.133) the problem if $N$ is very large: this involves the calculation of one determinant, $|\,\mathbf{C}\,|$, and the inversion of a large matrix **C**. The trace trick (1.136) deals with the determinant, but the matrix inversion still needs to be done, and we need to evaluate the logarithm of a matrix. [31]

## 1.11 Masked data on a sphere

In general we cannot use the entire sky for analysis of CMB data, the main obscuration being use to our Galaxy. So when we calculate the spherical harmonic transform of a field whose values are only known in part of the sky. We must take account of the mask when analysing the structure of the CMB in the Planck sky.

The mask, or window function, $W(\theta,\phi)$ is a complex-shaped region (Figure 1.1)and has to be described in terms of its spherical harmonics $w_{lm}$:

$$w_{lm} = \int_\Omega Y^*_{lm}(\theta,\phi)\; W(\theta,\phi)\, d\Omega. \tag{1.138}$$

The simplest mask would be $W(\theta,\phi) = 0$ in the entire masked region, and 1 elsewhere. This tends to generate artefacts because of the sharp edges of this function and so the practice is to make the transition of $W(\theta,\phi)$ from 0 to1 smooth.

what smoother should be used?

If we suppose that the spherical harmonic coefficients of the whole sky in the absence

[31] The the natural logarithm of a matrix **A** is the matrix **B** such that $e^{\mathbf{B}} = \mathbf{A}$, and we write $\mathbf{B} = \mathbf{ln}\mathbf{A}$. **B** exists if and only if **A** has an inverse. For a general non-singular matrix **A** with real elements the the elements of $\mathbf{B} = \mathbf{ln}\mathbf{A}$ could be complex numbers, in which case the conjugate of **B** is also $\mathbf{ln}\mathbf{A}$. However, if **A** is diagonal, then **B** is a diagonal matrix whose elements are simply the logarithms of the elements of **A**. Thus the way to compute $\mathbf{ln}\mathbf{A}$ is to first diagonalise **A** using the matrix **W** such that $\mathbf{E} = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$ is diagonal with elements that are the eigenvalues of **A**. Then $\mathbf{B} = \mathbf{ln}\mathbf{A} = \mathbf{W}\,(\mathbf{ln}\,\mathbf{E})\mathbf{W}^{-1}$.
The matrix logarithm satisfies $\mathbf{ln}\mathbf{I} = 0$, where **I** is the identity matrix, and $\mathbf{ln}\,(\mathbf{PQ}) = \mathbf{ln}\mathbf{P} + \mathbf{ln}\mathbf{Q}$.
Here, the logarithm of a matrix **A** is denoted by a boldface **ln** to emphasise that $\mathbf{ln}\mathbf{A}$ is itself a matrix. The result of the trace operator tr is a scalar, and so this is not boldfaced.

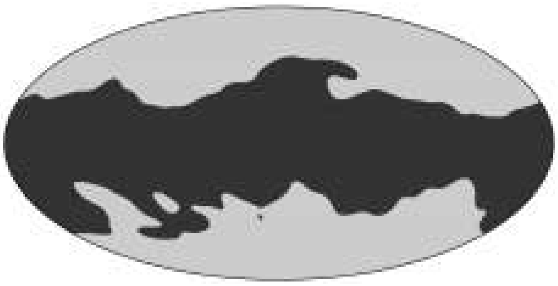

**Fig. 1.1** Mask G45 for the Planck sky. The mask is defined in terms of a threshold applied to the 353 Hz map. Sharp boundaries define the boundary of the dark excluded region. To use this mask it is first apodised, *i.e.* smoothed to soften the edges of the mask.. (From Planck Collaboration XV (2013, Figure B.1))

of a mask are $a_{lm}$, then the transform of the data as seen through the window $W(\theta, \phi)$ is (Hivon et al., 2002, Appendix A)

$$\hat{a}_{lm} = \sum_{l',m'} F(l', m', l, m)\, a_{l'm'} \tag{1.139}$$

where the function $F(l', m', l, m)$ is

$$F(l', m', l, m) = (-1)^m \sum_{l''m''} \sqrt{\frac{(2l''+1)(2l'+1)(2l+1)}{4\pi}} \begin{pmatrix} l'' & l' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l'' & l' & l \\ m'' & m' & -m \end{pmatrix} w_{l''m''} \tag{1.140}$$

the terms in parentheses being the Wigner $3j$-symbols (see equation 1.31).

The main reason to use the spherical harmonics to describe the entire sky is that they form a statistically independent set of modes describing the sky: the $Y_{lm}$ are an orthogonal set of basis functions on the sphere. This is identical to using Fourier analysis on a line or on a plane. We notice from equation (1.139) that the coefficients $\hat{a}_{lm}$ describing the masked sky are not independent:

$$\langle \hat{a}_{l'm'} \hat{a}_{l''m''} \rangle = \sum_{lm} F^*(l, m, l', m') F(l, m, l'', m'')\, C_l \tag{1.141}$$

where $C_l$ is defined as the covariance of the unmasked sky $a_{lm}$:

$$\langle a_{lm} a_{l'm'} \rangle = \delta_{ll'} \delta_{mm'}\, C_l \tag{1.142}$$

We note that for the unmasked sky, the $a_{lm}$ are mutually independent for all but those having the same indices. This is the consequence of the orthogonality of the $Y_{lm}$.

This is IN-PAINTING!! Check this - should be stated with care. Is it essential?

The goal is to invert equation (1.139) to find the $a_{l'm'}$ of the CMB in terms of the measured $\hat{a}_{lm}$. This is by no means easy but it is an essential step towards getting to the ultimate goal of CMB temperature anisotropy experiments: the power spectrum of the fluctuations (Hivon et. al. *loc. cit.*). Inevitably, it is the small-$l$ modes that will be most affected by the choice of mask and by the inaccuracies that may arise in recovering the "true" power spectrum.

The way to verify the reliability of the resultant inversion is to build accurate models of the sky that include the CMB itself and the foregrounds, and to use the various techniques to recover the original CMB map. It should be remembered that, however good our inversion might be, we cannot get around the cosmic variance problem: we only have the one realisation of out Universe to work with.

One way of dealing with this is to not apply a mask and simply do a separation of the map into independent components. We shall turn to that later.

## 1.12 Angular scales corresponding to $P_l(\cos\theta)$

There is no unique way of defining an angular scale on the sky corresponding to a given $l$-value.[32] At small $l$, $l = 0$ corresponds to the isotropic component, $l = 1$ corresponds to a dipole and $l = 2$ a quadrupole. This leads to the simplistic idea that the angular scale associated with a value of $l$ is simply

$$\theta = \frac{180}{l} \quad \text{degrees}, \qquad 1 \leq l < 10 \quad \text{or so} \tag{1.143}$$

The upper value of $l$ for which this makes intuitive sense is inevitably somewhat vague.

### 1.12.1 Jeans' rule relating $l$-value to length scale

There is an interesting remark made by Jeans ('Jeans' rule') that the plane wave wavelength associated with a spherical harmonic $Y_{lm}$ is approximately

**Jeans' rule**

$$\lambda \simeq \frac{2\pi}{(l+\frac{1}{2})}, \quad l \gg 1 \tag{1.144}$$

relating wavelength $\lambda$ to the $l$-value

This comes about by noticing that $Y_{lm}$ is a solution of the eigenvalue problem (1.19), which written in 2-dimensional $(x, y)$ coordinates is $w_{xx} + w_{yy} = -l(l+1)w$. Substituting $w = w_0 \exp[i(k_x x + k_y y)]$ leads to $k^2 = k_x^2 + k_y^2 = l(l+1)$. With this definition of $k$, the wavelength is $\lambda = 2\pi/k = 2\pi/\sqrt{l(l+1)}$. When expanded in a power series in $l$ this gives $\lambda = 2\pi/(l+\frac{1}{2}) + O(l^{-2})$. [33] This is Jeans' rule as given in equation (1.144). The restriction to $l \gg 1$ comes about because in using a plane wave Fourier representation on a small enough section of the sphere that the curvature effects are negligible.

[32] One reason for this is that the zero-curves of the $Y_{lm}(\theta, \phi)$ do not bound areas of roughly similar shape except at the smaller $l$-values. At large $l$ and small $m$ the shapes are highly elongated and do not have an evident sense of scale.

[33] This explanation is based on R.L. Parker's lecture notes `http://igppweb.ucsd.edu/~parker/SIO229/jeans.p.pdf` (possibly volatile)
Here we have used the convention $w = w_0 \exp[i(k_x x + k_y y)]$ that is used in cosmology. The wavelength associated with a wavenumber $k$ is the distance between successive peaks, *i.e.* $2\pi/k$.

In cosmology we tend to refer to the scale of a ’patch’ on the sky: a region included within the zero-contour of the field $w(x,y)$. The size of a ’patch’ is the scale of the region included within the zero-contour of the field $w(x,y)$ rather than the wavelength. In that case $\lambda_{pathch} = \pi/k$ and we get $\lambda_{patch} = \pi/(l+\frac{1}{2})$. See section 1.12 for a more complete discussion of how to interpret angular scales.

### 1.12.2 Patch size from the correlation function

However, this is not as obvious a choice at large $l$-values that describe small enough angular scales as to isolate a patch on the sky surrounding a given point. We have already seen the simple ‘Jeans rule’ (equation 1.144) for $l \gg 1$ which was based on a localised plane-wave analysis on the surface of the sphere.

If we think of the angle $\theta$ as being measured from a particular point on the sky, the logical scale of the patch defined by $P_l(\cos\theta)$ is the angle $\lambda$ at which $P_l(\cos\lambda) = 0$ for the first time. This can be motivated by considering the angular correlation function (1.110) of the temperature fluctuations. If there was only a single non-zero coefficient $C_l$ then the correlation function would be $\xi(\theta) \propto P_l(\cos\theta)$. The patch size would be defined by the first zero, $\theta_0$, of the correlation function. *i.e.* the first zero of $P_l(\cos\theta)$. It’s largely a matter of taste as to whether we would assign a size $\theta_0$ or $2\theta_0$ to the patch. Here we will use $\theta_0$ because that seems to be the standard convention in cosmology. Peebles (1980, equation 46.26) uses the value $\theta_0 = 3\pi/4l$ to describe the angular scale corresponding to $l$, while Jean’s rule (equation 1.144) argues for $2\pi/l$.

For large $l$ this is quite straightforward to calculate using the large-$l$ approximation for $P_l(\cos\theta)$. [34] The scale of the patch is then

$$\lambda_l = \theta_0 = \frac{3\pi}{4(l+\frac{1}{2})} = \frac{135}{l+\frac{1}{2}} \text{ degrees}, \quad l \gg 1 \tag{1.145}$$

Notwithstanding this argument, the scale (1.143) is in general use, and that is the convention we adopt throughout.

### 1.12.3 Sampling the correlation function

As a final remark on this topic we can ask how many $l$-values, $l_{max}$, are required to describe the correlations in a map of a given angular resolution or, conversely, how many $l$-values are probed by an experiment having a given angular resolution. We could simply use equation (1.145) to address this, but the following argument is perhaps somewhat more intuitive.

There are some $4 \times 10^4$ square degrees over the whole sky ($4\pi$ steradians) and so a map

[34] From Abramowitz and Stegun (1965, Equation 8.10.7):

$$P_\nu(\cos\theta) \simeq \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\sqrt{\frac{2}{\pi\sin\theta}}\cos\left[\left(\nu+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + O(\nu^{-1}), \quad \frac{1}{\nu} < \theta < \pi - \frac{1}{\nu}$$

The first zero of this occurs at $\theta_0$ where $\left(\nu+\frac{1}{2}\right)\theta_0 - \frac{\pi}{4} = \frac{\pi}{2}$, *i.e.* $\theta_0 \simeq 3\pi/4\nu$. If we need the approximate values of $P_\nu(\cos\theta)$ in the range of angles where the approximation is valid, the $\Gamma$-functions can be approximated by Stirling’s formula and then $\Gamma(\nu+1)/\Gamma(\nu+\frac{3}{2}) \simeq \nu^{-1/2}$.

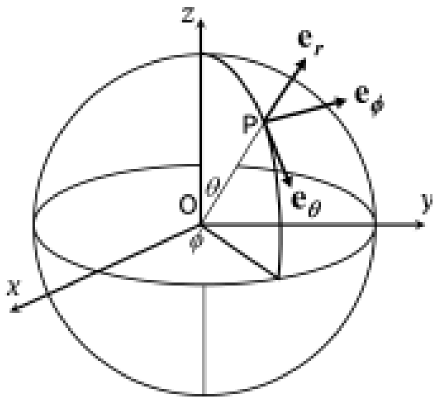


**Fig. 1.2** The spherical polar coordinate system $(r, \theta, \phi)$ with the unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ at a point on the surface. $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ are in the tangent plane and $\mathbf{e}_r$, is normal to that plane. The vectors point in the direction of increase in their coordinate values, and the triad is a right handed.

sampled at 1□° would have $\sim 4 \times 10^4$ independent pixels. If we set this to $l_{max}^2$, then for a 1□° sampling we would be going to $l_{max} \sim 200$. So, for example, the COBE satellite with its beam of $\sim 7^\circ$ would only provide information for $l \lesssim 30$ or so. To map structure to $l_{max} \sim 2000$ requires a resolution of $\sim 6'$.

The argument is simple, but why are there $l_{max}^2$ elements? For each $l$-value there are $(2l + 1)$ $a_{lm}$'s, and so if we have all values of $l \leq l_{max}$ we have $\sum_{l=0}^{l_{max}}(2l + 1) = l_{max}^2$ coefficients in all.

In general, in a given experiment different frequency channels provide different beam widths, the lower frequencies providing the lower resolutions. So not all frequencies cover the same range of $l$-values.

## 1.13 Complexification of fields on a sphere

There is a final manipulation that can be done: that is to rewrite the preceding as vectors having complex number values in axes that are in the space of complex numbers. This is a key step towards describing the polarisation map in terms of spinors, and so we touch on it briefly here. Spinors, like tensors, have covariant and contravariant components, and in addition have the property of complex conjugation since they work on the field of complex numbers. Here the focus will be on how this is used to describe the polarisation maps and its relationship with the spherical harmonics.

Here we follow Phinney and Burridge (1973) since that is one of the clearest statements of this process.[35]

[35] The didactic review by Torres Del Castillo (2007) shows how this development is related to spinors.

### 1.13.1 Complex basis vectors

First we note that we can combine our orthogonal unit basis vectors $(\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r)$ to form a new set of complex vectors on the ground field of complex numbers (see Figure 1.2 for the definition of our coordinate system). We denote this new basis by $(\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+)$ and we choose to use the following definition:

$$\mathbf{e}_- = \frac{1}{\sqrt{2}}(\mathbf{e}_\theta - i\mathbf{e}_\phi), \quad \mathbf{e}_0 = \mathbf{e}_r \quad \mathbf{e}_+ = -\frac{1}{\sqrt{2}}(\mathbf{e}_\theta + i\mathbf{e}_\phi) \tag{1.146}$$

The reason for this particular choice of basis vectors is that they are eigenvectors of the matrix describing infinitesimal rotations about the $Oz$-axis. Any vector of complex numbers in 3-space can be represented as a complex number weighted sum of these basis vectors.[36]

We can easily verify that

$$\mathbf{e}_+ \cdot \mathbf{e}_+ = \mathbf{e}_- \cdot \mathbf{e}_- = \mathbf{e}_0 \cdot \mathbf{e}_0 = 1, \qquad \mathbf{e}_0 \cdot \mathbf{e}_+ = \mathbf{e}_0 \cdot \mathbf{e}_- = \mathbf{e}_+ \cdot \mathbf{e}_- = 0 \tag{1.148}$$

The $(\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+)$ form an orthonormal triad of complex vectors in terms of which we can express any vector $\mathbf{u}$ as having components we denote by $(u^-, u^0, u^+)$. The issue to address now is how to relate $(u^-, u^0, u^+)$ to the real space components $(u_r, u_\theta, u_\phi)$.[37]

### 1.13.2 Transforming between vector bases

We seek mathematical descriptions of a 3-dimensional space in terms of two related vector spaces: one is a real vector space and the other a complex vector space. We need to construct the relationship between them and to show how scalar, vector and tensor descriptions of physical quantities in the two spaces are derived from one another.

**Information 1.13.2**

**Use of Greek and Latin indices**
It is necessary to distinguish the indices on vectors in two spaces: the real space and the complex space. We have chosen to use Latin lower case indices for the components in the real space and Greek letters for components in the complex ground space. As always, repeated indices imply summation unless otherwise stated.

The link between the real space view and the complex-space view of vectors and tensors

[36] Vectors over the space of complex numbers $\mathbb{C}$ do not behave in quite the same way as the familiar vectors defined on the space of real numbers $\mathbb{R}$. In order that the length of a complex vector be positive or zero, we need to define the (Euclidean) scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i^*$, where the $a_i$ and the $b_i$ are the complex components of the representation of the vectors in some basis. (This is described as being "linear in the first and conjugate n the second") We then have $\mathbf{a} \cdot \mathbf{a} \geq 0$, with equality if and only if $\mathbf{a} = \mathbf{0}$. We can easily show that, for $\lambda \in \mathbb{C}$,

$$(\mathbf{a} \cdot \mathbf{b})^* = \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda^*(\mathbf{a} \cdot \mathbf{b}), \quad (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda \mathbf{a} \cdot \mathbf{b}. \tag{1.147}$$

Note that we could have defined $\mathbf{a} \cdot \mathbf{b} = \sum_i a_i^* b_i$: this simply is an alternative convention. When the vectors are regarded as column vectors, the scalar product is the Hermitian product $\mathbf{a}^\dagger \mathbf{b}$ where $\mathbf{a}^\dagger = (\mathbf{a}^*)^T$.
It does not matter which convention is adopted, but the two conventions should not be mixed!

[37] We write the components of the vector with upper (contravariant) indices so as be able to write $\mathbf{u} = u^\alpha \mathbf{e}_\alpha$.

is provided by the unitary matrix

$$\mathbf{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{1.149}$$

Denoting the complex conjugate of the transpose of $\mathbf{C}$ by $\mathbf{C}^\dagger$, the fact of $\mathbf{C}$ being unitary gives us $\mathbf{C}^\dagger\mathbf{C} = \mathbf{C}\mathbf{C}^\dagger = \mathbf{I}$, as can be verified.[38] We can appreciate the role of $\mathbf{C}$ (equation 1.149) if we re-index the real space basis as $\mathbf{e}_i = (\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r)$ and the complex space basis as $\mathbf{e}_\alpha = (\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+)$ so that we have

$$\mathbf{e}_\alpha = C_{i\alpha}\mathbf{e}_i, \tag{1.150}$$

The implied summation is on the index $i$ which labels the *rows* of $\mathbf{C}$.

We use $\mathbf{C}$ to relate the components of the contravariant vector $u^\alpha = (u^-, u^0, u^+)$ in the basis $(\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+)$ to the components of the real-space vector $\mathbf{u} = (u_\theta, u_\phi, u_r)$ in the basis $(\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r)$:

$$u^\alpha = \sum_{i=1}^{3} C^\dagger_{\alpha\, i} u_i, \quad u_i = (u_\theta, u_\phi, u_r) \tag{1.151}$$

The vector field $\mathbf{u}$, expressed in spherical coordinates by components $\mathbf{u} = (u_\theta, u_\phi, u_r)$, has the contravariant representation in the complex basis as

$$u^- = \frac{1}{\sqrt{2}}(u_\theta + iu_\phi), \quad u^0 = u_r \quad u^+ = -\frac{1}{\sqrt{2}}(u_\theta - iu_\phi) \tag{1.152}$$

The transformation of a tensor quantity $t_{ij}$ in the real-space to this complex space is achieved via the double matrix multiplication

$$t^{\mu\nu} = C_{\mu\, i} C_{\nu,j}\; t_{ij} \tag{1.153}$$

and, in particular, the Kronecker symbol $\delta_{ij}$ transforms as

$$\boldsymbol{\epsilon}_{\mu\nu} = C_{\mu\, i} C_{\nu\, j}\; \delta_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.154}$$

Note the Greek indices on $\boldsymbol{\epsilon}_{\mu\nu}$: this matrix operates on quantities in the complex space.

In effect, we can think of the matrix $\boldsymbol{\epsilon}_{\mu\nu}$ as a metric on our space of complex vectors. Just as the length of a vector $\mathbf{v} = \{v_a\}$ in real Euclidean space can be written $\|\mathbf{v}\| = v_i v_j \delta_{ij}$ we shall define the norm of a vector $\mathbf{u} = \{u^\alpha\}$ in our complexified space as $u^\alpha u^\beta \epsilon_{\alpha\beta}$.

The raising and lowering of indices between the covariant and contravariant components of a vector is achieved via the matrix $\epsilon_{\mu\nu}$:

$$u_\alpha = \epsilon_{\alpha\beta}\, u^\beta \tag{1.155}$$

[38] The arrangement of the rows and columns of $\mathbf{C}$ depends on choice of the ordering of the components in the vector. Here we are arranging the components of a vector $\mathbf{u}$ in the real space in the order $(u_\theta, u_\phi, u_r)$ and the components of the corresponding vector in the complex space in the order $(u^-, u^0, u^+)$. This follows the usage of Phinney and Burridge (1973).

With this we have, for the contravariant and covariant versions of any **v** in this complex basis:

$$v^a = (v^+, v^0, v^-), \quad v_a = (-v^-, v^0, -v^+), \tag{1.156}$$

The scalar product of two vectors **u** and **v** is then

$$\mathbf{u} \cdot \mathbf{v} = u^\mu v^\nu \epsilon_{\mu\nu} = u_a v^a = u^0 v^0 - u^+ v^- - u^- v^+ \tag{1.157}$$

By simple substitution we can verify that this scalar product $\mathbf{u} \cdot \mathbf{v} = u_r v_r + u_\theta v_\theta + u_\phi v_\phi$, as would be expected.

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