Gas Dynamics

A Supplement to “Precision Cosmology”

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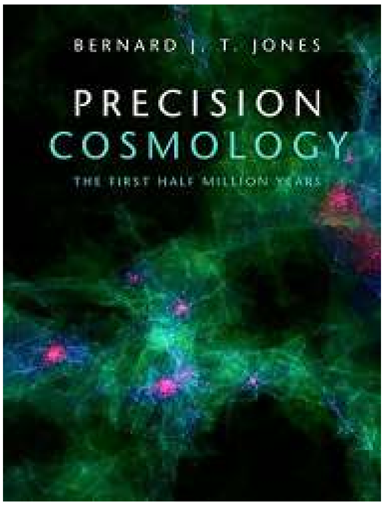

The fundamental equations of gas/fluid flow consist of dynamical equations, an equation describing the conservation of material, and an energy equation. Under conditions where there is heat exchange and dissipation these equations become more complex.

This is one of a set of Supplementary Notes and Chapters to “Precision Cosmology”.
Some of these Supplements might have been a chapter in the book itself, but were regarded either as being somewhat more specialised than the material elsewhere in the book, or somewhat tangential to the main subject matter.
The are mostly early drafts and have not been fully proof-read.
Please send comments on errors or ambiguities to “`PrecisionCosmology(at)gmail.com`”.

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1.1 The state of a one-component gas

A state gas will be described by a number of variables, each defined at a position $\mathbf{x}$ and time t:

- Density $\rho(\mathbf{x}, t)$
- Pressure $p(\mathbf{x}, t)$
- Temperature $T(\mathbf{x}, t)$
- Velocity $\mathbf{v}(\mathbf{x}, t)$

In addition to this there are boundary conditions that provide constraints that these variables must satisfy.

1.1.1 The thermodynamic state of the gas

The three scalars will be inter-related by an equation defining the thermodynamic state of the gas. Consider a volume V of the gas, containing N moles of the gas, so that the density is N/V. The simplest such equation is the *ideal gas law* which states that

$$pV = NRT \tag{1.1}$$

where $R = 8.3145$ Joules K^{-1}mole^{-1} is the *gas constant* for a unit mass of gas. The *internal energy* of this volume of gas is

$$U = NC_V T \tag{1.2}$$

where the constant C_V is the *specific heat at constant volume* of the gas. C_V is a measure of how much the temperature of the gas changes, ΔT, when an amount heat ΔQ is added while keeping the volume fixed:

$$C_V = \left(\frac{\Delta Q}{\Delta T}\right)_V \tag{1.3}$$

The *specific heat at constant pressure*, C_P, is likewise defined as a measure of how much the temperature of the gas changes, ΔT, when an amount of heat ΔQ is added while keeping the pressure fixed:

$$C_V = \left(\frac{\Delta Q}{\Delta T}\right)_P \tag{1.4}$$

The relationship between C_P and C_V is

$$C_P = C_V + R \tag{1.5}$$

where R is the gas constant, and it is customary to define a quantity γ by

$$\gamma = \frac{C_P}{C_V}, \tag{1.6}$$

the *ratio of the specific heats*. An alternative definition of γ can be used when the specific heats are not constants:

$$pV = (\gamma - 1)U \tag{1.7}$$

The *number of degrees of freedom* of a molecule of gas, f, is $f = 3$ for a monatomic gas, $f = 5$ for a rigid diatomic molecule , and $f = 6$ for a rigid molecule of arbitrary shape. When C_V is a constant

$$\gamma = \frac{2}{f} + 1 = \frac{5}{3} \quad \text{for a monatomic gas} \tag{1.8}$$

The state of the gas is subject to the *First and Second Laws of Thermodynamics*. The First law sates that energy is conserved. Thus if we add heat ΔQ to a volume V of gas, and the volume V does an amount of work ΔW on its surroundings, then the internal energy, ΔU of the volume will change and

$$\Delta Q = \Delta U + \Delta W \tag{1.9}$$

In the case when the change in the state of the gas is associated with a change ΔV in volume we have $\Delta W = p\Delta V$.

A change wherein $\Delta Q = 0$ is called an *adiabatic* change:

$$\Delta U = -\Delta W, \quad \text{adiabatic} \tag{1.10}$$

The second law tells us that there exists a property of the gas, its *entropy*, S. If of the state of the gas is changed from one equilibrium state to another by a making small change in volume, the change in the internal energy U is given by

$$dU = TdS - pdV \tag{1.11}$$

An adiabatic change has zero change in entropy and then $dU = -pdV$.

Consider an adiabatic change in the state of the gas. We can rearrange equation (1.11) to read $dU/T = -(p/T)dV$. Using dU from equation (1.2) and $p/T = NR/V$ from equation (1.1), we obtain

$$\frac{dT}{T} = -\frac{R}{C_V}\frac{dV}{V} = -(\gamma - 1)\frac{dV}{V} \tag{1.12}$$

from which follow the various well known relations for an adiabatic change to s simple gas:

$$TV^{\gamma-1} = \text{const.}, \quad pV^{\gamma} = \text{const.} \tag{1.13}$$

1.2 The non-relativistic gas dynamics equations

Three sets of equations are needed: an equation representing the conservation of the number of particles, and equation reflecting the conservation of energy, and an equation for the conservation of momentum of the fluid elements as they move under the influence of external forces.

1.2.1 Derivatives following the fluid motion

First we start with a technical point of some importance. When talking about the time dependence of some physical property of a fluid element, we can sit at one place and watch the fluid go by and draw a graph of that property as a function of time. So if we were measuring the density at that fixed point we would think of it as $\rho(\mathbf{x}, t)$ where $\mathbf{x}$ is the fixed point of observation and t is the time. Alternatively we can sit on the particle itself and measure the density at that particle as the fluid is carried around by the general flow. If we then think of the density as $\rho(\mathbf{x}, t)$ in this sense, the position $\mathbf{x}$ is in fact time dependent: it moves with the fluid and it is not a fixed point.

It is easy to relate these two. In a short period of time δt the particle of fluid moves from its original position at $\mathbf{r}$ to a new position at $\mathbf{r} + \mathbf{v}\delta t$. Put $\delta\mathbf{r} = \mathbf{v}\delta t$, the displacement of the particle in time δt. The change in density in shifting from one point to the next is due to two contributions: the change with time in the density field and the change in location of the measuring point:

$$\delta\rho = \frac{\partial\rho}{\partial t}\delta t + \delta\mathbf{r}\cdot\mathbf{v}\,\rho \tag{1.14}$$

The second term on the right describes the component of the density gradient in the direction $\delta\mathbf{r}$. Since $\delta\mathbf{r} = \mathbf{v}\delta t$, this is

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho \tag{1.15}$$

The derivative on the left hand side is interpreted as the *time derivative following the motion*, as compared with the first term on the left with is the time derivative at a given (fixed) place. There are a couple of technical points to make about this. This derivative is sometimes referred to as the *material derivative* or the *substantial derivative*.

The notation is perhaps a little unfortunate. Some authors would write this as

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho$$

if only to emphasise that the left hand side is not really a simple derivative.

The other technical point is that the acceleration of the fluid felt by an element of fluid moving with the flow is

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{1.16}$$

The brackets in the last term are important since $(\mathbf{v}\cdot\nabla)\mathbf{v} \neq \mathbf{v}(\nabla\cdot\mathbf{v})$. In fact

$$(\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{v}(\nabla\cdot\mathbf{v}) = \mathbf{v}\times(\nabla\times\mathbf{v}) \tag{1.17}$$

which is an important vector relationship in fluid dynamics.

1.2.2 Conservation of particles

Consider an elemental volume V in which the density of the gas is $\rho(\mathbf{x}, t)$ and in which the gas velocity is $\mathbf{v}(\mathbf{x}, t)$. The the mass of gas in the volume is $\int \rho\, dV$ and the flux of gas through the boundary of the volume is $\oint_S \rho\mathbf{v}\, dS$, where S is the bounding surface of V. The rate of change of mass within V must be the flux of mass through the surface S of V:

$$\frac{\partial}{\partial t}\int \rho dV = -\int \rho\mathbf{v}\; dS \;= -\int \nabla\cdot(\rho\mathbf{v})dV \tag{1.18}$$

by Gauss' theorem. In the limit $V \to 0$ this gives the *equation of continuity*:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{1.19}$$

1.2.3 Conservation of momentum

As the fluid moves, its acceleration is proportional to the forces acting on it. Sitting on a particle of fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{f} \tag{1.20}$$

where f are the external forces acting at $\mathbf{x}$. This form of the equation assumes that there is neither viscosity nor heat transport.

1.2.4 Thermal energy

The *thermal energy equation* is an equation for the rate of change of the internal energy e of the gas, which in the case of an ideal gas is simply $e = C_V T$. In the absence of dissipative processes, e will change due to the 'pdV' work done by the pressure forces.

$$\rho\frac{de}{dt} = -p\nabla\cdot\mathbf{v} \tag{1.21}$$

1.2.5 Viscosity and thermal conduction

The previous development assumes an ideal gas in which there is no internal transfer of energy through viscosity or thermal conduction. These processes do not affect the conservation of mass, but they do affect the momentum equation and the energy equation.

The complexity of the equations increases and it is best to write them in a vector format in which the velocity, relative to some (non-rotating) Cartesian frame of reference has components

$$\mathbf{x} = \{x_i\} = (x_1, x_2, x_3)$$
$$\mathbf{v} = \{v_i\} = (v_1, v_2, v_3)$$

It is, moreover, useful to write equations for the components of the vectors and to invoke the summation convention whereby repeated indices imply summation:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i \stackrel{def}{=} a_i b_i \tag{1.22}$$

We also need the standard Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.23}$$

The full equations can be summarised as follows:

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} \tag{1.24}$$

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{v}\right)\right] + \rho f_i \tag{1.25}$$

$$\rho \frac{dE}{dt} = -p\nabla \cdot \mathbf{v} + \mu\left[\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 - \frac{2}{3}(\nabla \cdot \mathbf{v})^2\right] + \nabla \cdot (\kappa \nabla kT) \tag{1.26}$$

Here μ is the coefficient of shear viscosity, κ is the thermal conductivity, and k is the Boltzmann constant. We assume here that the coefficient of *bulk viscosity* is zero: this is the case for a normal monatomic normal, but may not be the case for molecular gases, nor for relativistic gases.

In the everyday case where atomic mean free paths, l_c, are small compared with the density gradients, we can approximate these by simple expressions

$$\mu \simeq \rho l_c \overline{v}, \quad \kappa \simeq n l_c \overline{v}, \quad l_c \simeq (n\sigma)^{-1}, \quad \overline{v} = \sqrt{kT/m} \tag{1.27}$$

where σ is the cross section for collision of the gas particles and $\overline{v}$ is the mean thermal velocity of the particles which have mass m. The accurate calculation of these dissipation coefficients comes from approximations to the Boltzmann equation, expanding the collision term in powers of the particle mean free path. They are applicable when mean free paths are very short compared with the length scales of gradients appearing in the problem.

In the case when the mean free path, l_c becomes comparable with the scale l of the density gradients, we can modify the viscosity with a *flux limiter*. The basic idea here is that in the limit of infinite mean free path, the viscosity and thermal conductivity are effectively zero and so an interpolation formula is needed that takes the transport coefficients as given in equation (1.27) to zero in the limit $l \to 0$. A simple example would be

$$\mu \to \frac{1}{1+\frac{l_c^2}{l^2}}\,\mu \to \begin{cases} 0 & \text{as } l_c \gg l \\ \mu & \text{as } l_c \ll l \end{cases}, \quad l \simeq \left|\frac{\nabla\rho}{\rho}\right|^{-1} \tag{1.28}$$

This is a very simplistic interpolation formula, but it turns out that it is quite appropriate in the study of the Universe around the period of recombination when the dominant transport process is Thomson scattering.

Flux limiters play a key role in numerical gas dynamic simulations.

1.3 Special Relativity

The key gas dynamic equations (1.19), (1.20), and (1.21) are not Lorentz invariant. In other words, changing to another frame of reference using the Lorentz transformation changes these equations. The general Lorentz transformation between inertial frames $S(\mathbf{r}, t)$ and $S'(\mathbf{r}', t')$ where S' moves at velocity $\mathbf{v}$ relative to S is[1]

$$\mathbf{r}' = \gamma(\mathbf{r}^\dagger - \mathbf{v}t) \tag{1.29}$$

$$t' = \gamma(t - [\mathbf{v}\cdot\mathbf{r})/c^2] \tag{1.30}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{1.31}$$

$$\mathbf{r}^\dagger = \mathbf{r}/\gamma \tag{1.32}$$

This may look fearfully complex, that is one of the main reasons for using tensors rather than vectors to do relativity. Not surprisingly, the resultant equations are more complex than their no-relativistic counterparts, but they nevertheless are recognisable. The equations of continuity and motion are respectively:

$$\gamma\frac{d\rho}{dt} + \left(\rho + \frac{p}{c^2}\right)\left[\frac{\partial\gamma}{\partial t} + \nabla\cdot(\gamma\mathbf{v})\right] = 0 \tag{1.33}$$

$$\gamma^2\left(\rho + \frac{p}{c^2}\right)\frac{d\mathbf{v}}{dt} = -\nabla p - \frac{\mathbf{v}}{c^2}\frac{\partial p}{\partial t} \tag{1.34}$$

The relativistic γ-factors are perhaps not surprising, but the appearance of the pressure alongside the density in the $\rho + p/c^2$ term needs comment. It means that if you perform an experiment to measure the the fluid density in your local inertial frame than you will have to subtract p/c^2 from the measurement in order to recover the rest-mass density.

[1] Note that the symbol γ is used here because this is the conventional symbol for the Lorentz factor. Of course the symbol γ is also used to denote the ratio of specific heats of a gas and appears throughout the hydrodynamic equations. However, the context should easily indicate which is intended.

For non-relativistic velocities, $v \ll c$ these equation reduce to

$$\frac{d\rho}{dt} + \left(\rho + \frac{p}{c^2}\right)\nabla\cdot\mathbf{v} = 0 \tag{1.35}$$

$$\left(\rho + \frac{p}{c^2}\right)\frac{d\mathbf{v}}{dt} = -\nabla p \tag{1.36}$$

These are the equations that underlie the pseudo-Newtonian derivation of the cosmological equations.

Need some GR relativistic discussion to discuss the $\rho + 3p)$ as a source of gravity

1.4 An analogy - 2-dimensional fluid flow

The goal here is to consider the properties of a 2-dimensional vector field. We focus on the specific example of the velocity field **u** of a fluid flow in 2-dimensions.

We show firstly that we can decompose any vector field into rotational and irrotational parts. The former correspond physically to the vorticity of the flow, while the latter correspond to the shear and expansion of the flow. In the case of a 2-dimensional flow the vorticity and shear matrices are closely related to the 2×2 spin matrices.

Pursuing the analogy further we then restrict ourselves to steady incompressible flow and complexify the flow field via the Cauchy-Riemann equations. This closely parallels the complexification of the polarisation components Q and U in the case of electromagnetic fields. We finish by writing the complexified fluid equations in terms of a complex potential which is an analytic function in the complex plane.

1.4.1 Vector fields: rotation and shear

Here we follow Batchelor (2000, §2.4). A general result in the theory of vector fields tells us that any vector field can be decomposed into the sum of two vectors:

'Electric' and 'Magnetic' components of a general vector field

$$\mathbf{u} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\times\mathbf{B} \tag{1.37}$$

This is immediately reminiscent of thinking of the electromagnetic field as being made up of two components, one of which, **B**, is divergence-free and the other of which, **E**, is derivable from a potential. This splitting up of a vector or tensor field into a divergence-free and a rotation-free part is generally thought of splitting a field into its 'electric' and 'magnetic' components, even when the field has nothing to do with electromagnetism.[2]

The component $\mathbf{u}_e = \boldsymbol{\nabla}\varphi$ of the flow is clearly irrotational, but can have non-zero divergence. The other component, $\mathbf{u}_v = \boldsymbol{\nabla}\times\mathbf{B}$, must have zero divergence. The vorticity is

[2] In general relativity, the Weyl tensor is split into an 'electric', E_{ab} and a 'magnetic', H_{ab} component.

$\omega = \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$, and if $\nabla \cdot \mathbf{B} = 0$ everywhere,[3] we have the equation for the vorticity

$$\omega = -\nabla^2 \mathbf{B} \tag{1.38}$$

1.4.2 General 2-dimensional flow field

In the case of a 2-dimensional flow where $\mathbf{u} = \nabla \times \mathbf{B}$ we can write the shear and vorticity directly in terms of the derivatives of the vector potential $\mathbf{B}$. Writing the components of the vector $\mathbf{B}$ as $(0, 0, \psi)$, so that $\mathbf{u} = (-\psi_{,2}, \psi_{,1}, 0)$, we can write the derivatives $u_{a,b}$ of the components u_a of the velocity in a 2×2 matrix:

$$\mathbf{\Psi} = \{u_{a,b}\} = \begin{pmatrix} -\psi_{,12} & \psi_{,11} \\ -\psi_{,22} & \psi_{,12} \end{pmatrix} \tag{1.39}$$

The comma, as in $\psi_{,a}$, indicates the partial derivative with respect to x_a.

This matrix can be split into its symmetric σ_{ab} and skew-symmetric ω_{ab} parts:

$$\sigma_{ab} = \begin{pmatrix} -\psi_{12} & \frac{1}{2}(\psi_{11} - \psi_{22}) \\ \frac{1}{2}(\psi_{11} - \psi_{22}) & \psi_{21} \end{pmatrix} \tag{1.40}$$

$$\omega_{ab} = \begin{pmatrix} 0 & \frac{1}{2}(\psi_{11} + \psi_{22}) \\ -\frac{1}{2}(\psi_{11} + \psi_{22}) & 0 \end{pmatrix} \tag{1.41}$$

We see that the shear σ_{ab} is trace-free. This is consistent with the flow velocity having zero divergence, $\nabla \cdot \mathbf{u} = 0$, as was mandated by our assumption that $\mathbf{u} = \nabla \times \mathbf{B}$ and hence divergence free. We also see that the magnitude of the vorticity ω_{ab} is $\omega = 2\omega_{ab}\omega_{ba} = -\nabla^2\psi$ (using the summation convention to indicate sum over a and b). Alternatively we can verify that the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ has $\omega_3 = \epsilon_{3jk}\omega_{kj} = -\nabla^2\psi$ (ϵ_{ijk} is the completely anti-symmetric 3-index permutation symbol), which is consistent with ().

1.4.3 2-dimensional steady incompressible flow

For a steady irrotational incompressible flow in 2-dimensions we can define a velocity potential φ and a stream function ψ (**?**, §10) which satisfy Laplace's equation in two dimensions and so allow us to easily generate a large number of solutions for complex flow patterns. Despite its simplicity, this mechanism of discovering important solutions to the equations of fluid motion proved invaluable at a time when computers did not exist.

Consider a simple (non-relativistic) steady incompressible fluid flow in two (Cartesian) dimensions (x_1, x_2). The flow velocity $\mathbf{u}$ then has components $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$. The assumption that the flow is steady implies conservation of mass in any element of volume, which in turn tells us that there exists a *stream function* ψ such that $\nabla^2\psi = 0$. For a two dimensional flow in 3-dimensions we can write $\mathbf{u} = \nabla \times \mathbf{B}$ where the vector field $\mathbf{B} = (0, 0, \psi)$.

With the assumption that the flow is irrotational, there exists a potential φ such that the

[3] Batchelor *loc. cit.* shows that $\omega = -\nabla^2\mathbf{B}$ is consistent with $\nabla \cdot \mathbf{B} = 0$ whenever there is no normal component of vorticity on any boundary of the fluid, or in the case of an infinite fluid.

velocity field $\mathbf{u} = \boldsymbol{\nabla}\varphi$. The condition of incompressibility tells us further that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and so $\nabla^2\varphi = 0$.

This leads to the set of equations

$$\nabla^2\varphi = 0, \quad \nabla^2\psi = 0, \quad \boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi = 0 \tag{1.42}$$

$$u_1 = \partial_1\varphi = \partial_2\psi, \quad u_2 = \partial_2\varphi = -\partial_1\psi \tag{1.43}$$

where we have used the notation $\partial_a = \partial/\partial x_a,\ a = 1,2$. We note that $\partial_1 u_1 + \partial_2 u_2 = 0$, which is the continuity equation for this scenario and $\partial_1 u_2 - \partial_2 u_1 = 0$ which is the conditions that the flow is irrotational. The last of equations (1.42) tells us that the flow lines and the stream lines are orthogonal except where these gradients are zero.

The equations (1.43), written out in full, are

Cauchy Riemann equations

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}, \quad \frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1} \tag{1.44}$$

These are called the *Cauchy Riemann equations*.[4] Functions φ and ψ satisfying these equations automatically satisfy the Laplace equation.

1.5 Complexification of 2-dimensional fluid flow

The *complexification* of these equations is achieved with the transformations to new, complex, variables:

$$z = x + iy \tag{1.45}$$

$$u = u_1 + iu_2 \tag{1.46}$$

$$\omega(z) = \varphi + i\psi \tag{1.47}$$

$$\tag{1.48}$$

in which case the flow equations become

$$\frac{d\omega}{dz} = u_1 - iu_2 \tag{1.49}$$

So it is relatively simple to generate steady incompressible 2-dimensional flows: just pick a function $\omega(z)$ and differentiate it.[5] We have to be aware of the boundaries imposed on the solution by the requirement that the Cauchy Riemann equations only hold where $\omega(z)$ is analytic. However, the fact that the equations are linear means that we can superpose solutions having different boundary conditions. We also have the possibility of morphing

[4] These were discussed in the context of fluid flow by both Euler and d'Alembert long before Cauchy and Riemann.

[5] If the function $\omega(z) = \varphi + i\psi$ is analytic in some domain $\mathcal{D}$, then φ and ψ have continuous first derivatives in $\mathcal{D}$ and satisfy the Cauchy Riemann equations. The converse is also true.

one flow into another by making a transformation of the variable z into another complex variable.[6]

1.5.1 A simple example of 2-dimensional fluid flow

A trivial example of such a flow is

$$\omega(z) = q\, e^{-i\alpha}\, z \tag{1.50}$$

where q and α are real constants. Using (x, y) to denote the coordinates, we hen have

$$\varphi + i\psi = q(\cos\alpha - i\sin\alpha)(x + iy) \tag{1.51}$$

with which the velocity potentials and stream function are given by

$$\varphi = q(x\cos\alpha + y\sin\alpha), \quad \psi = q(y\cos\alpha - x\sin\alpha) \tag{1.52}$$

with velocity components

$$u_x = q\cos\alpha, \quad u_y = q\sin\alpha. \tag{1.53}$$

The flow is uniform flow with speed q, the stream lines making an angle α with the x-axis. The textbook of Kundu and Cohen (2010, Ch. 4.) has many in-depth examples of such flows.

The shear tensor, σ_{ab}, and vorticity tensor, ω_{ab}, are defined by

$$\sigma_{ab} = \frac{1}{2}(\partial_b u_a + \partial_a u_b), \quad \omega_{ab} = \frac{1}{2}(\partial_b u_a - \partial_a u_b), \quad a, b = 1, 2 \tag{1.54}$$

With this it is easy to verify that the magnitude of the shear, σ and vorticity, ω are

$$\sigma = \frac{1}{2}(\Psi_{11} - \Psi_{22}), \quad \omega = \frac{1}{2}\epsilon_{ab3}(u_{a,b} - u_{b,a}) = -\frac{1}{2}\nabla^2\Psi \tag{1.55}$$

[6] Any function of two variables $f(x, y)$ can be complexified by replacing (x, y) by $z = (x + iy$ and $z^* = x - iy$. This leads to a function $f(z, z^*)$. The situation here is that we are only considering the very special case when $f(z, z*)$ can be expressed as a function $\omega(z)$ of the single variable z.

References

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