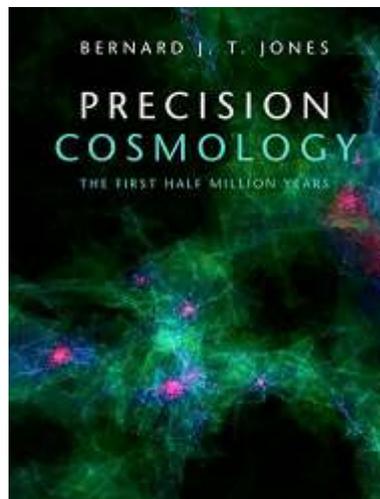


Kinetic Theory

A Supplement to “Precision Cosmology”

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Kinetic theory studies the motion of systems of interacting particles at a statistical level.

This is one of a set of Supplementary Notes and Chapters to “Precision Cosmology”. Some of these Supplements might have been a chapter in the book itself, but were regarded either as being somewhat more specialised than the material elsewhere in the book, or somewhat tangential to the main subject matter.

The are mostly early drafts and have not been fully proof-read.

Please send comments on errors or ambiguities to “PrecisionCosmology(at)gmail.com”.

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1.1 The kinetic theory of gases

The *kinetic theory of gases* is the microscopic, atomic level, description of gases that underlies the fluid model that is used in cosmology. Generally speaking, we can simply take classical laboratory fluid mechanics (or gas dynamics) straight over into the theory of relativity and use that to address cosmological problems. However, there are some important problems where modelling the micro-physics that takes place is of importance: by dealing with the micro-physics we avoid the approximations that go into deriving the full fluid/gas equations. One goal of kinetic theory is to put the thermodynamics and dynamics of fluid motions onto a firm physics footing.

The development of the kinetic theory has been going on since the beginning of the 19th century, but its history stretches back to the time of Boyle, Hooke and Newton around 1660¹, or even to Torricelli (Galileo's student) and Pascal at the beginning of the 17th century. See Brush and Hall (2003) for an annotated collection of papers dating from that time.

The theory of the motion of fluids has been taken up by renowned mathematicians, physicists and engineers ever since Newton discussed the nature of viscous forces and described fluids whose viscosity did not change with the rate of flow, a *Newtonian fluid*. The basic equations governing inviscid fluid flow were known as long ago as 1757 when written down by Euler (1757)². Euler did not provide an energy equation: that was introduced in a simple form by Laplace in 1816. The introduction of viscosity into the equations came with the work of Navier in 1822³ who wrote down the equations we now know as the Navier-Stokes equations.

The Navier-Stokes equations described the dissipative process of viscosity that had been missing from the idealised perfect fluid of Euler. Those equations contained explicit terms for the action of viscosity: there was no understanding of the physical mechanism for the viscosity and so it had to be determined empirically. The microscopic description of

¹ Newton (1726, first edition 1687) in his *Principia* had observed that Boyle's Law could be reproduced if inter-particle forces were repulsive and scaled as $1/r$.

² Euler was perhaps one of the greatest mathematicians of all time. One of his teachers was Johann Bernoulli who had earlier been professor of mathematics at the University of Groningen. When living in St. Petersburg Euler became close friends with Johann's son, Daniel Bernoulli, who became famous for his work on fluid mechanics and on probability and statistics. Daniel's *Hydrodynamica* was plagiarised and backdated by Johann for his own *Hydraulica*. Johann and Daniel did not get on well together.

³ Navier was a student of Fourier and in 1830 succeeded Cauchy as professor of calculus and mechanics at the École Polytechnique after Cauchy exiled himself from France for political reasons.

gas and fluid flows came with the fundamental development of kinetic theory of gases by Boltzmann in 1872. Boltzmann had the theory, but the theoretical determination of the coefficient of viscosity and other transport processes based on kinetic theory did not come until the work of Chapman and Enskog in the early 20th century.

Short biography: J.J. Waterston - the unacknowledged pioneer

In 1846 J.J. Waterston published an abstract of his work on the kinetic theory of gases in the *Proceedings of the Royal Society*. In that paper he made many landmark advances: he was, for example, the first to suggest that the temperature of a gas was due to random motions and was proportional to the square of the thermal velocity. The paper had been submitted the previous year and it seems that it was ignored, but not actually rejected (the referee, apparently a distinguished expert in the field, commented that “... the paper is nonsense, unfit even for reading before the Society”). Despite repeated attempts to draw attention to this work over the next three decades by Waterston himself, the paper was only published in 1892 in the *Philosophical Transactions of the Royal Society* (Waterston, 1892) after its “discovery” by Lord Rayleigh in the Society archive in 1891. Waterston, after a long period of bitterness directed at the scientific community, disappeared in 1883, and was presumed to have committed suicide.

Rayleigh wrote an important preface to the paper in which he remarks that “... it marks an immense advance in the direction of the now generally received theory.” and goes on to say that “The omission to publish it at the time was a misfortune, which probably retarded the development of the subject by ten or fifteen years”. However, he does stop short of apology in remarking that “... a young author who believes himself capable of great things would usually do well to secure the favourable recognition of the scientific world by work whose scope is limited, and whose value is easily judged, before embarking on greater flights”. See Brush (1961) for a more detailed account.

Boltzmann (unlike his predecessor Waterston (1892)) had built his theory on the firm foundations of the work in mechanics of Hamilton and Liouville, and in doing so created one of the great pillars of modern science that must rank alongside the achievements of Einstein and Bohr.

1.1.1 The phase space distribution function

In the kinetic theory of gases the movement of the particles making up the gas is divided into two parts: a systemic motion which we shall denote by $\hat{\mathbf{v}}$ and a chaotic motion \mathbf{v} relative to $\hat{\mathbf{v}}$. These represent a localised average of the velocities of the particles and the local deviations from that average. Precisely how that average is taken is a matter of making further assumptions about the local state of the gas, taking into account, for example, of whether the gas is locally in thermodynamic equilibrium.

The number of particles, dN , in a volume $d^3\mathbf{x}$ having velocities in a range of values $d^3\mathbf{v}$ is given by a distribution function $f(\mathbf{x}, \mathbf{v}, t)$:

$$dN = f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{x} d^3\mathbf{v} \quad (1.1)$$

The function f is normalised so that the integral over all the position and velocity space occupied by the fluid is the total particle number N . In this simplistic model, $f(\mathbf{x}, \mathbf{v}, t)$ depends only on the positions and velocities of the particles that make up the fluid. In a more realistic case f would also depend on the molecular state of the gas, and atomic physics and radiative processes that might take place. If more than one particle species is present there will be a distribution function f_α for each of the species.

It should be noted that equation (1.1) is a *single particle distribution function*: it describes the probability of finding a single particle in an element of volume in phase space. A more general statistical distribution would need to describe the joint distributions of particles in which particle positions and velocities are correlated on microscopic scales because of specific modes of interaction (*e.g.* quantum liquids).

Here we focus on the macroscopic dynamics of gases in which binary collisions take place, and whose state can be described by a single particle distribution function such as (1.1). The gas may be made up of multiple components, *e.g.* protons, electrons and photons, and each component may be described by its own single particle distribution function. How such a gas behaves may depend critically on the micro-physics of the collision processes.

1.1.2 Macroscopic quantities

The local density, n , and flow velocity, $\hat{\mathbf{v}}$, at a point \mathbf{x} are then given by integrals over the velocity-space of the distribution function at the place \mathbf{x} :

$$n(\mathbf{x}, t) = \int_{\mathbf{v}} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}, \quad \hat{\mathbf{v}}(\mathbf{x}, t) = \frac{1}{n} \int_{\mathbf{v}} \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \quad (1.2)$$

where the integrals are over the velocity space occupied by the fluid. These are the zeroth and first statistical moments of the local velocity distribution function. The second moment

$$p_{\mu\nu} = \int_{\mathbf{v}} m(v_\mu - \hat{v}_\mu)(v_\nu - \hat{v}_\nu) f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}, \quad \mu, \nu = 1, 2, 3 \quad (1.3)$$

is the pressure tensor, and we define a temperature via the trace of this tensor:

$$\frac{3}{2}nk_B T = \int_{\mathbf{v}} \frac{1}{2}(\mathbf{v} - \hat{\mathbf{v}})^2 f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \quad (1.4)$$

This definition is used even if the gas is not in equilibrium. The third moment

$$Q_{\mu\nu\sigma} = \int_{\mathbf{v}} m(v_\mu - \hat{v}_\mu)(v_\nu - \hat{v}_\nu)(v_\sigma - \hat{v}_\sigma) f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \quad (1.5)$$

describes the heat flux. From the tensor $Q_{\mu\nu\sigma}$ we can construct a vector quantity

$$\mathbf{Q} = \int_{\mathbf{v}} \frac{1}{2}m(\mathbf{v} - \hat{\mathbf{v}})^2(\mathbf{v} - \hat{\mathbf{v}}) f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \quad (1.6)$$

which we can understand as being a velocity vector weighting of the kinetic energy in random motions $\frac{1}{2}m(\mathbf{v} - \hat{\mathbf{v}})^2$. In other words a thermal energy flux.

From equation (1.3) and (1.4) we have that the temperature and pressure are related by

$$p = \frac{1}{3} \sum_{\mu=1}^3 p_{\mu\mu} = nk_B T \quad (1.7)$$

which is the ideal gas law. What this means is that this level of kinetic theory is inadequate to deal with anything other than ideal gases.

The macroscopic quantities $n, \mathbf{v}, p_{\mu\nu}, T$ are the observables of fluid flows and we need to develop equations that inter-relate these and equations that govern their evolution in physical gas/fluid flows. These latter equations are the *Navier-Stokes* equations which take a variety of different forms depending not only on the statistical distribution function of the

particles, but also on the nature of the fluid. The macroscopic variables ultimately depend on the microscopic nature of the fluid: the nature of the particles that make up the fluid and their mutual interactions.

The task is to find an equation for $f(\mathbf{x}, \mathbf{v}, t)$ in terms of the micro-physics that underlies the flow. This problem was first solved by Boltzmann in 1872 who wrote down the key equation that now bears his name. The macroscopic equations for the flow velocity had been known since 1822 but they could not be derived from more fundamental principles. Using that equation to derive generic equations for fluid motion was a problem that remained until the work of Chapman and Enskog in the 1930's.

1.2 The Boltzmann equation

We begin this discussion by looking at the statistical distribution of particles, $f(\mathbf{x}, \mathbf{v}, t)$, in a gas where there are no collisions between its constituent particles. In 1833, Hamilton had reformulated the Lagrangian view of mechanics so as to describe the state of a dynamical system in terms that regarded both positions and momenta as independent variables linked by the equations of motion.⁴ The space of Hamiltonian dynamics was the 6-dimensional phase space of the positions \mathbf{x} and their conjugate momenta \mathbf{p} , as opposed to the 3-dimensional space of Lagrange's dynamics. By 1838, Liouville had proved the fundamental theorem for systems of particles that is known as the *Liouville theorem*.

Liouville wrote an equation describing the evolution of a dynamical ensemble of points in phase space (the *Liouville equation*). From this he proved his theorem expressing the fact that the local phase space density in the vicinity of a point moving through phase space is a constant (Arnold, 1980, section 16). Another way of describing this is to say that along the motion of any fluid element in phase space, the phase space density is a constant.⁵

1.2.1 The collisionless Boltzmann equation

On the basis of Liouville's theorem Boltzmann was able to write down an equation for the evolution of the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ under the assumption that the particles making up the system did not suffer collisions with other particles:

Collisionless Boltzmann equation

$$\frac{\mathcal{D}f}{\mathcal{D}t} \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \mathbf{g} \cdot \nabla_{\mathbf{v}} f = 0 \quad (1.8)$$

⁴ Almost a century later the Hamiltonian view was to provide an important basis for formulating quantum mechanics.

⁵ We could say that the flow in phase space is 'locally' incompressible, but this should not be taken to imply that the density is independent of position as in the case of an incompressible liquid in 3-dimensions. By 'locally' we would mean 'in the limit of infinitesimally small phase space volume, since the statement would be manifestly false for any finite volume element because of *phase space mixing* (unlike in an incompressible liquid). Such a *coarse-graining* of the phase space is inevitable in simulating the Boltzmann equation. See Arad and Lynden-Bell (2005) and Arad and Johansson (2005) for more discussion of this last point.

This is the *collisionless Boltzmann equation*⁶. In this equation the gradients $\nabla_{\mathbf{r}}$ and $\nabla_{\mathbf{v}}$ are gradients with respect to the coordinates \mathbf{r} and \mathbf{v} respectively and \mathbf{g} is the external force exerted per unit mass on a fluid element. If the force were due to an electromagnetic field the expression for \mathbf{g} would be the Lorentz force on a particle $\mathbf{g} = q\mathbf{E} + \mathbf{v} \times \mathbf{B}/c$ and we would have the Vlasov equation for a plasma.

The differential operator $\mathcal{D}/\mathcal{D}t$ is referred to as the Liouville operator.

The external force field, $\mathbf{f}f$, is generally set to zero but there are important circumstances where it has to be considered. It plays a vital role in plasma physics where the charged particles are moving in electromagnetic fields. When the Boltzmann equation is used to do fluid dynamics modelling in rotating systems, such as the Earth's atmosphere, there are centrifugal and Coriolis forces to take into account. Most importantly in the present context, in general relativity there are the forces due to the space-time geometry that have to be included.

1.2.2 The Poincarè-Jeans Theorem

For systems that are stationary in the sense that the distribution function $f(\mathbf{x}, \mathbf{v})$ does not depend on time, there is an important theorem due to Poincarè and Jeans that tells us that solutions of the stationary collisionless Boltzmann equation are functions of the invariants of the motion.

In the collisionless situation the particles move on orbits in the force field \mathbf{g} with no deflection away from the orbits. The equations of motion of one of the particles are simply

$$\frac{d\mathbf{v}}{dt} = \mathbf{g}, \quad \frac{d\mathbf{r}}{dt} = \mathbf{v} \quad (1.9)$$

These equations, when integrated, have solutions that involve six constants of integration, $\alpha = \{\alpha_1, \dots, \alpha_6\}$:

$$\mathbf{v} = \mathbf{v}(\alpha, t), \quad \mathbf{r} = \mathbf{r}(\alpha, t) \quad (1.10)$$

In principle, we can solve these equations for the α as a function of phase space location (\mathbf{r}, \mathbf{v}) and time:

$$\alpha_i = \alpha_i(\mathbf{r}, \mathbf{v}, t), \quad i = 1, \dots, 6. \quad (1.11)$$

The fundamental theorem due to Poincaré (1911) and to Jeans (1915) tells us that any function of the α_i 's, $f(\alpha_1, \dots, \alpha_6)$, is a solution of the collisionless Boltzmann equation.

This is easily verified by putting $f(\alpha_1, \dots, \alpha_6)$ into equation (1.8):

$$\sum_{i=1}^6 \frac{\partial f}{\partial \alpha_i} \left(\frac{\partial \alpha_i}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \alpha_i + \mathbf{g} \cdot \nabla_{\mathbf{v}} \alpha_i \right) = \sum_{i=1}^6 \frac{\partial f}{\partial \alpha_i} \frac{d\alpha_i}{dt} = 0, \quad (1.12)$$

since, by definition, the α_i are constants of integration. They are in fact constants of the

⁶ In the 1938 Vlasov applied this equation to the dynamics of plasmas by including inter-particle electromagnetic forces in the force term \mathbf{g} . The collisionless Boltzmann equation is also referred to as the *Vlasov equation*, even outside of plasma physics. See Bertschinger (1995) for a more complete discussion.

motion and so they are referred to as *integrals of the motion*. The theorem is called *Jean's Theorem*. This argument was first given in the context of the Vlasov equation (the collisionless Boltzmann equation when the force law is the Lorentz force) by Watson (1956) which has led to significant developments in plasma physics based on particle orbits.

The result is also of considerable importance in stellar dynamics (Binney and Tremaine, 2008, Ch. 4.). If the force is not time dependent, then the energy integral is one of the α_i , and we have further integrals if the potential of the force has symmetries. It might be thought that the existence of a solution $f(\alpha_1, \dots, \alpha_6)$ imposes one constraint on the motion of a particle in phase space by confining the orbit to a 5-dimensional hyperplane and that each additional integral reduces the dimensionality of the space occupied by the orbits by one. The situation is, however, not that simple.

Lynden-Bell (1962a,b) showed that not all integrals have the same status in terms of the constraints they impose on the phase space orbits. Some integrals, *isolating integrals* do constrain the orbit to a sub-volume of the possible phase space while others, the *non-isolating integrals*, impose no such constraint and allow the orbit to fill the entire space. The most obvious integral, the energy integral for the system, does impose one constraint: it is an isolating integral and confines the orbit to a 5-dimensional space. If the system has a symmetry, such as spatial axi-symmetry, then that is associated with an isolating integral and in this case the integral is the angular momentum. Thus a spherically symmetric solution admits four isolating integrals (the energy and the three components of the angular momentum).

1.2.3 Macroscopic equations - no dissipation

Using the collisionless Boltzmann equation (1.8) and the definitions (1.2) we can deduce a macroscopic conservation law.

$$\frac{\mathcal{D}}{\mathcal{D}t} \int f(\mathbf{x}, \mathbf{v}, t) d^3 \mathbf{v} = \int \frac{\partial f}{\partial t} d^3 \mathbf{v} + \int (\mathbf{v} \cdot \nabla_{\mathbf{x}}) f d^3 \mathbf{v} + \int (\mathbf{g} \cdot \nabla_{\mathbf{v}}) f d^3 \mathbf{v} \quad (1.13)$$

The terms in this equation are easily simplified:

$$\int \frac{\partial f}{\partial t} d^3 \mathbf{v} = \frac{\partial}{\partial t} \int f d^3 \mathbf{v} = \frac{\partial n}{\partial t} \quad (1.14)$$

$$\int (\mathbf{v} \cdot \nabla_{\mathbf{x}}) f d^3 \mathbf{v} = \nabla \cdot \int \mathbf{v} f d^3 \mathbf{v} = \nabla \cdot (n\mathbf{v}) \quad (1.15)$$

$$\int (\mathbf{g} \cdot \nabla_{\mathbf{v}}) f d^3 \mathbf{v} = \int \nabla \cdot (\mathbf{g}f) d^3 \mathbf{v} = 0. \quad (1.16)$$

The last of these follows provided the force \mathbf{g} does not depend on velocity. Collecting these up and putting $\rho(\mathbf{x}, t) = n(\mathbf{x}, t)m$, where m is the mass per particle, we get

Collisionless Boltzmann zeroth moment - continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.17)$$

which is the familiar equation of continuity.

To get the macroscopic momentum equation we use a similar strategy and operate on $(\mathbf{v}f)$ with the Liouville operator:

$$\frac{\mathcal{D}}{\mathcal{D}t} \int \mathbf{v}f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = \frac{\partial}{\partial t} \int (\mathbf{v}f) d^3\mathbf{v} + \int \mathbf{v}(\mathbf{v} \cdot \nabla_{\mathbf{x}})f d^3\mathbf{v} + \int (\mathbf{g} \cdot \nabla_{\mathbf{v}})(\mathbf{v}f) d^3\mathbf{v} \quad (1.18)$$

Evaluating the integrals in this equation is not quite as easy as the previous one. In the case when the force \mathbf{g} does not depend on velocity⁷ and after substituting $\rho(\mathbf{x}, t) = n(\mathbf{x}, t)m$ as before, we get

Collisionless Boltzmann first moment - momentum equation

$$\rho \left(\frac{\partial}{\partial t} + v_\nu \frac{\partial}{\partial x_\nu} \right) v_\mu = \rho g_\mu - \frac{1}{\rho} \frac{\partial}{\partial x_\nu} P_{\mu\nu} \quad (1.22)$$

$$\text{where } P_{\mu\nu} = \int (v_\mu - \hat{v}_\mu)(v_\nu - \hat{v}_\nu) f d^3\mathbf{v} \quad (1.23)$$

$P_{\mu\nu}$ is the *pressure tensor*.

We shall see that if the collision mean free path is short compared with the gradients, then the pressure term $P_{\mu\nu}$ will be isotropic.

Finally there is an energy conservation equation that comes from operating with the Liouville operator on $(|\mathbf{v}|^2 f)$. The calculation is little different from the preceding one, though somewhat more tedious. The result, using equation (1.4), is

Collisionless Boltzmann second moment - energy equation

$$\frac{3}{2} n \left(\frac{\partial}{\partial t} + v_\nu \frac{\partial}{\partial x_\nu} \right) k_B T = -p_{\mu\nu} \frac{\partial v_\mu}{\partial x_\nu} - \nabla \cdot \mathbf{Q} \quad (1.24)$$

This delivers an important message: to be able to make use of the full equations for the dynamics of a gas, we need to be able to evaluate the quantities $p_{\mu\nu}$ and \mathbf{Q} . To do that we need to know what the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ is and hence we need to deal with the micro-physics aspects of the gas.

⁷ Dealing in turn with each term on the right hand side of equation (1.18):

$$\frac{\partial}{\partial t} \int (\mathbf{v}f) d^3\mathbf{v} = \frac{\partial}{\partial t} (mn\hat{\mathbf{v}}) \quad (1.19)$$

$$\begin{aligned} \int v_\mu \left(v_\nu \frac{\partial}{\partial x_\nu} \right) f d^3\mathbf{v} &= \frac{\partial}{\partial x_\mu} \int v_\mu v_\nu f d^3\mathbf{v} \\ &= \frac{\partial}{\partial x_\mu} \int (v_\mu - \hat{v}_\mu)(v_\nu - \hat{v}_\nu) f d^3\mathbf{v} + \frac{\partial}{\partial x_\mu} \left(\hat{v}_\mu \hat{v}_\nu \int f d^3\mathbf{v} \right) \end{aligned}$$

$$\therefore m \int v_\mu \left(v_\nu \frac{\partial}{\partial x_\nu} \right) f d^3\mathbf{v} = \frac{\partial P_{\mu\nu}}{\partial x_\mu} + \frac{\partial}{\partial x_\mu} (mn\hat{v}_\mu \hat{v}_\nu) \quad (1.20)$$

$$\int \left(g_\nu \frac{\partial}{\partial v_\nu} \right) (f v_\mu) d^3\mathbf{v} = - \int g_\nu \frac{\partial}{\partial v_\nu} (f v_\mu) d^3\mathbf{v} = -ng_\mu \quad (1.21)$$

The required expression comes from bringing together the right hand sides of the numbered equations.

1.2.4 Boltzmann's equation with collisions

The importance of the no-collisions requirement was that collisions could change the momentum of a particle and so move it out of its local phase space volume, thus rendering the Liouville theorem on conservation of phase space density inapplicable. However, in our every day experience it is the collision dominated liquids and gases that are the main subject of our interest. How do we go from the microscopic notion that gases and liquids are collections of atoms, molecules and other particles in a state of random motion to our view of the macroscopic flow of liquids and gases? The answer, at least for dilute gases, is given by the Boltzmann equation with a collision term which describes the underlying micro-physics.

Boltzmann's great step was to provide the equation governing the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in the presence of collisions. This is the *Boltzmann equation*:

Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \mathbf{g} \cdot \nabla_{\mathbf{v}} f = J, \quad J = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \quad (1.25)$$

The term J is the so-called *collision term* which describes the rate at which particles are scattered in and out of an elemental volume due to their mutual collisions. Writing an equation for the collision term requires detailed knowledge of the physical processes at the atomic or molecular level or a viable model for these processes.⁸

From the point of view of modelling ideal fluid flow, there are four important developments regarding the Boltzmann equation that have emerged during the past century:

- The Chapman Enskog method (Chapman and Cowling, 1939)
- The BGK model (Bhatnagar et al., 1954).
- The Lattice Boltzmann equation (McNamara and Zanetti, 1988)
- The relativistic Boltzmann equation

We shall discuss these briefly in the following sections. The details can be found in Huang (1989, Ch.3 and Ch.4).

⁸ An important example of 'knowing the physics' arises when the gas under consideration consists of fermions or of bosons: there are quantum factors appearing in the collision term that ultimately yield the Fermi-Dirac and Bose-Einstein distributions for ensembles of such particles. The classical gas, as we shall show, leads to the Maxwell-Boltzmann equilibrium distribution.

1.3 The interface with thermodynamics

1.3.1 Molecular chaos

In an important insight into the physical processes, Boltzmann (1872) provided an expression for the collision term expressing it in terms of the cross-sections for the microscopic processes (see equation (1.39) below). The essence of Boltzmann's collision term lay in his assumption of *molecular chaos* which allowed a fundamental simplification.

What was important is how often, in any elemental volume $d^3\mathbf{x}d^3\mathbf{v}$ of phase space, two particles would come together in the collision. This was expressed as a probability that was made to depend on the joint distribution of the particle positions and velocities, $f(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)$, where the subscripts '1' and '2' refer to the two particles involved in the collision. The simplifying assumption was the assumption that the particles involved in the collision were statistically independent:

Molecular chaos:

$$f(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f(\mathbf{x}_1, \mathbf{v}_1, t) f(\mathbf{x}_2, \mathbf{v}_2, t) \quad (1.26)$$

Statistics had entered into physics. This asserted that the particles entering the collision were uncorrelated, and, importantly, it implied that on leaving the collision the particles were still uncorrelated, despite the rules for the conservation of energy and momentum which govern the post-collision particle properties. The assumption of molecular chaos was insisting that these correlations would disappear very quickly before further collisions of those particle could occur. With this Boltzmann could bring in the concept of *detailed balance* in which the collision process was time-reversible and local equilibrium could prevail. Boltzmann's collision term, despite its complexity, has worked very well in addressing the flows of dilute gases where only binary collisions contribute.⁹

Modelling liquids and solids in terms of the underlying microscopic processes is far more difficult. A number of schemes have been developed for going beyond the simple binary Boltzmann collision term, notably the BBGKY hierarchy of equations. These work by substituting better models for the correlations expressed in equation (1.26).

1.3.2 Controversy: entropy and the H -theorem

Perhaps Boltzmann's greatest, and certainly his most controversial contribution, came with the celebrated H -theorem which provided a measure of the irreversibility of the

⁹ Experimental molecular chaos was reported for the first time by Baxter and Olafsen (2007) in a 'granular gas'. The system ran in a steady state with a Gaussian velocity distribution that was consistent with Maxwell-Boltzmann statistics, but was out of equilibrium. The important conclusion was that the velocities of the gas particles were uncorrelated, as required by the Molecular Chaos hypothesis. Earlier experiments had shown that non-Gaussian velocity distributions were associated with non-zero particle velocity correlations, and hence the absence of Molecular Chaos.

evolution of systems of particles. This measure was directly related to the distribution function $f(\mathbf{x}, \mathbf{v}, t)$:

$$-k_B S = H = \int f(\mathbf{x}, \mathbf{v}, t) \ln f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{x} d^3\mathbf{v} \quad (1.27)$$

where S is the entropy and k_B is *Boltzmann's constant*. Boltzmann was able to prove that H had to satisfy

Boltzmann's H -theorem

$$\frac{dH}{dt} \leq 0 \quad (1.28)$$

at those times when the state of the gas is described by the molecular chaos assumption.

Remarkably, this result was completely independent of the nature of the collision term. The theorem comes from Hamilton's formulation of mechanics and Liouville's insight into the incompressible nature of (\mathbf{x}, \mathbf{v}) phase space, the *Liouville equation*. Boltzmann had proved what was already known - that the entropy of an isolated system could only increase. But he had proved it using physics that was time reversible.

Why was this controversial? Simply because it derived a very general theorem on the irreversibility of systems of particles based on equations that were themselves microscopically time reversible. The debate started shortly after the publication of the H -theorem in 1872. Loschmidt (1876), a colleague of Boltzmann, pointed out that in the evolution of a system from a time t_0 to a later time t_1 , while according to the H -theorem the entropy was increasing, one could simply reverse the velocities of all the particles at t_1 and, because the underlying dynamics is time reversal invariant, the state at time t_0 would be restored. Hence H would have to increase on the way back from t_1 to t_0 in order to precisely restore the initial state. We apparently have a paradox. The arguments still persist to this day.

Expand on
orthodox response

The "orthodox" response to this 'paradox' is given in Huang (1989, §4.2.). The key to the argument lies in the phrase just after equation (1.27): "... at those times when the state of the gas is described by the molecular chaos assumption".

The problem is that the state of molecular chaos is an idealised model that cannot in fact exist. ter Haar (1955), in his excellent discussion of this issue, provides the following thought experiment. Place a population of hard spheres that are distributed randomly on a surface perpendicular to the x -axis, parallel to which is moving a beam of point particles that scatter off the spheres. The result is a population of point particles that, as a consequence of the scattering, are moving in isotropically and randomly distributed directions. Now reverse the paths of the particles. They bounce off the spheres and form a perfectly organised beam. So even for this simple system the conditions of molecular chaos are not at all satisfied.

This suggests that the assumption of molecular chaos is too idealised to properly model the question of entropy and is part of the problem. Notwithstanding that, the collisional Boltzmann model is evidently perfectly adequate for discussing the simpler aspects of dissipative phenomena, and so we would be reluctant to abandon it.

1.3.3 Entropy I

Less than a decade before Boltzmann wrote his equation, Clausius, in 1846, wrote a paper formalising the notion of *entropy*. The steam engines and other engines of the industrial revolution were not 100% efficient and the question had been asked as to why not all their energy was going into work, where was the ‘lost’ energy going? This was an important question because making engines more efficient was financially advantageous.

The early 19th century had seen a considerable development of thermodynamics, notably with the work of Carnot (1824)¹⁰ and his *caloric theory of heat* in which heat was regarded being propagated by some fictitious fluid called *caloric*.¹¹

In 1854 Clausius introduced a new physical quantity which he referred to as the ‘equivalence value’ of a system and which in 1864 he renamed the *entropy*, labelled with the symbol S . If you add a quantity of heat ΔQ to a system at temperature T , the entropy changes by

$$\Delta S = \frac{\Delta Q}{T} \quad (1.29)$$

ΔS measures the irreversibility of the process of transferring heat within a system at temperature T . With this came Clausius’ statement that “heat can never pass from a colder to a warmer body without some other change”. This was the first precise statement of the second law of thermodynamics.

Clausius saw entropy as a measure of processes occurring at the molecular level, but did not formalise that. That was left to Boltzmann who in 1877 wrote down that

$$S = k_B \ln \Omega \quad (1.30)$$

where Ω is the number of micro-states of a system that would be consistent with the macroscopic state of the system. See Section ?? on the Multinomial distribution for more on this. With the H -theorem it looked as though Boltzmann had provided a theory explaining why the entropy of an isolated system could never decrease. This raised the question, and with it the paradoxes, of how using only time-reversible micro-physics, we could build a system that had a manifest *arrow of time*.

The connection between Boltzmann’s H and Clausius’ S comes from considering the specific case of a collisionless gas where S can be derived from thermodynamic principles for which H can be calculated from the Maxwell-Boltzmann distribution. If the gas density

¹⁰ There were two Carnot’s, the father Lazare, and the son, Sadi. Lazare worked extensively and published on the phenomenon of heat loss in machines. He was the first to state that there could be no practical realisation of a perpetual motion machine. This was perhaps the first statement of the second law of thermodynamics. Lazare died in 1823 and his son Sadi took up his father’s work and introduced what is now called the *Carnot cycle*, the ideal heat engine, and with it the concept of *caloric*, which was the stuff lost by machines while doing work. The ideal heat engine could not be realised in practise because of the loss of caloric.

¹¹ The alternative theory to the caloric theory was the relatively unknown kinetic theory of Herepath (1820). Herepath’s paper had been submitted to the Royal Society in 1820 but was rejected by Davy, perhaps because Herepath had suggested, for the first time, that there might be a minimum absolute temperature. However, it was published in a different journal in 1821 and brought to the attention of an unenthusiastic scientific community by Joule in 1848. A similar thing happened with a kinetic theory that was developed by Waterston (submitted 1850, published posthumously 1892). Kinetic Theory, as we now know it, was published in its present form by Boltzmann (1862).

is ρ and the temperature is T then both S and H are proportional to $\rho(\frac{3}{2} \ln T - \ln \rho)$ (see ter Haar (1995, §§1.6,1.7)).

At the time Boltzmann published his work little or nothing was generally known about statistics other than what was relevant to gambling. It was not until the 1890's that Pearson introduced histograms and correlations! In fact Boltzmann himself had appreciated towards the end of the 19th century that the interpretation of the H -theorem was related to statistics and what we now call information theory. In viewing the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ as a given probability density Boltzmann could be said to have failed to distinguish between the distribution and the physical realisation of a sample set drawn from that distribution.¹²

1.3.4 Entropy II

There are deeper questions surrounding the H -theorem, and indeed the entire concept of entropy. Some of these issues, with possible resolutions, are discussed in an essay by Gull (1989) (who refers to the H -theorem as “That awful H -theorem”) and by Jaynes (1965) who probes the relationship between the Boltzmann and Gibbs definitions of entropy. See Section ?? for the relationship of entropy and information.

This is supposed to
be on Information
Theory Entropy

1.4 Collisions

1.4.1 Elastic collisions

Modelling and computing the collision term J in equation (1.25) is an entire branch of kinetic theory and gas dynamics in itself. In the simplest scenario we consider the elastic collision between a pair of atoms or molecules in which prior to the collision they have velocities $\mathbf{v}_1, \mathbf{v}_2$ and after the collision they emerge with velocities $\mathbf{v}'_1, \mathbf{v}'_2$. Note that we are excluding considerations of excitation of internal degrees of freedom.

The probability of such a collision taking place is given in terms of the centre of mass frame differential cross section which will be written as

$$d\sigma(\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2) = \frac{d\sigma}{d\Omega} d\Omega \quad (1.31)$$

The cross section will depend on the relative directions of the incoming and of the outgoing particles. The quantity Ω represents the angle between the relative incoming $\mathbf{v}_2 - \mathbf{v}_1$ and outgoing $\mathbf{v}'_2 - \mathbf{v}'_1$ directions.

Consider a particle in a small volume $d^3\mathbf{r}_1$ moving with velocity \mathbf{v}_1 in some range $d^3\mathbf{v}_1$. We want to know the rate at which it collides with particles in the volume moving with velocity \mathbf{v}_2 . The rate at which particles moving at velocity \mathbf{v}_2 collide with the particle moving at \mathbf{v}_1 is proportional to two factors: the flux of such particles having a given value of the relative velocity $|\mathbf{v}_2 - \mathbf{v}_1|$ and the cross section for those collisions. The contribution

¹² Boltzmann did introduce the notion of an *ergodic system* in 1887 (?), at which point, with the work of Gibbs, statistical mechanics was born. See the overview of ter Haar (1955).

from the flux is $|\mathbf{v}_2 - \mathbf{v}_1| f(\mathbf{r}, \mathbf{v}, t) d^3 \mathbf{v}_2$ and the probability of such a collision occurring is the differential cross section multiplied by an element of solid angle, *i.e.* $d\Omega$. So

$$\text{collision rate (single target)} = |\mathbf{v}_2 - \mathbf{v}_1| f(\mathbf{r}, \mathbf{v}_2, t) d^3 \mathbf{v}_2 \frac{d\sigma}{d\Omega} d\Omega \quad (1.32)$$

Since a fraction $f(\mathbf{r}, \mathbf{v}_1, t)$ of target particles have velocity \mathbf{v}_1 the total rate of collisions is

$$\text{collision rate (total)} = f(\mathbf{r}, \mathbf{v}_1, t) \int_{\mathbf{v}_2} \int_{\Omega} |\mathbf{v}_2 - \mathbf{v}_1| f(\mathbf{r}, \mathbf{v}_2, t) \frac{d\sigma}{d\Omega} d\Omega d^3 \mathbf{v}_2 \quad (1.33)$$

and there is a similar equation for the rate at which a particle in $d^3 \mathbf{r}$ is scattered into an element of velocity $d^3 \mathbf{v}$ at \mathbf{v} :

$$\text{emergent rate (total)} = f(\mathbf{r}, \mathbf{v}', t) \int_{\mathbf{v}'_2} \int_{\Omega} |\mathbf{v}'_2 - \mathbf{v}'_1| f(\mathbf{r}, \mathbf{v}'_2, t) \frac{d\sigma'}{d\Omega} d\Omega d^3 \mathbf{v}'_2 \quad (1.34)$$

1.4.2 Detailed balance

In the centre of mass frame of two equal mass colliding particles

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}'_1 + \mathbf{v}'_2 \quad (1.35)$$

and the elastic scattering condition tells us that the magnitude of the relative velocity, v_R , of the two particles before and after collision is unchanged

$$v_R = |\mathbf{v}_1 - \mathbf{v}_2| = |\mathbf{v}'_1 - \mathbf{v}'_2| \quad (1.36)$$

Elastic collisions have the property of being time reversal invariant. By time reversal invariant we mean that the ingoing collisions and outgoing products of the collision would look the same if the velocities were reversed: we could not tell the difference between a movie of a collision played wither forwards or backwards. This is called *detailed balance* and is expressed as:

$$d\sigma(\mathbf{v}'_1, \mathbf{v}'_2 \rightarrow \mathbf{v}_1, \mathbf{v}_2) d^3 \mathbf{v}'_1 d^3 \mathbf{v}'_2 = d\sigma(\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2) d^3 \mathbf{v}_1 d^3 \mathbf{v}_2 \quad (1.37)$$

Without this assumption the system of particles would not be in local equilibrium.

With the assumption that the scattering process is time reversal invariant we can simplify the collision integral:¹³

Elastic scattering collision integral

$$J(\mathbf{r}, \mathbf{v}_1, t) = \int_{\mathbf{v}_2} \int_{\Omega} [f(\mathbf{v}'_1) f(\mathbf{v}'_2) - f(\mathbf{v}_1) f(\mathbf{v}_2)] v_R d\sigma(\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2) d^3 \mathbf{v}_2 \quad (1.39)$$

¹³ In writing this equation there is an implicit assumption that the particles entering into a collision are independent: their motion is not correlated and their velocities are not correlated with position. Thus the probability of having two particles entering an element of volume $d^3 \mathbf{r}$ with velocities \mathbf{v}_1 and \mathbf{v}_2 lying in small ranges $d^3 \mathbf{v}_1$ and $d^3 \mathbf{v}_2$ is

$$f(\mathbf{r}, \mathbf{v}_1, t) d^3 \mathbf{r} d^3 \mathbf{v}_1 \times f(\mathbf{r}, \mathbf{v}_2, t) d^3 \mathbf{r} d^3 \mathbf{v}_2 \quad (1.38)$$

This is an essential aspect of Boltzmann's notion of *molecular chaos*. Without this assumption we enter into the more complex arena of the BBGKY hierarchy.

The differential cross-section, equation (1.31), relates $d\sigma$ to $d\Omega$. Here, the outgoing velocities $\mathbf{v}'_1, \mathbf{v}'_2$ are considered to be functions of the incoming $\mathbf{v}_1, \mathbf{v}_2$ and the scattering angle, so the integration is over all incoming velocities \mathbf{v}_2 and scattering angles.¹⁴

1.4.3 The H -theorem

Boltzmann recognised that the term $f(\mathbf{v}'_1)f(\mathbf{v}'_2) - f(\mathbf{v}_1)f(\mathbf{v}_2)$ in the collision integral (1.39) would vanish when $f = f^0$. He went on to define a function $H[f]$ of the distribution function by

$$H(t) = \int_{\mathbf{v}} f(\mathbf{v}, t) \log f(\mathbf{v}, t) d^3\mathbf{v} \quad (1.40)$$

and showed that if $f(\mathbf{v}, t)$ is a solution of

$$\frac{\partial f(\mathbf{v}', t)}{\partial t} = \int_{\mathbf{v}_2} \int_{\Omega} [f(\mathbf{v}'_1)f(\mathbf{v}'_2) - f(\mathbf{v}_1)f(\mathbf{v}_2)] v_R d\sigma(\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2) d^3\mathbf{v}_2 \quad (1.41)$$

then

Boltzmann's H -Theorem

$$\frac{dH}{dt} \leq 0, \quad \text{if } f(\mathbf{v}, t) \text{ satisfies the Boltzmann equation (1.25).} \quad (1.42)$$

Equality is achieved only for the Maxwell-Boltzmann distribution.

The condition $f(\mathbf{v}'_1)f(\mathbf{v}'_2) - f(\mathbf{v}_1)f(\mathbf{v}_2) = 0$ can be written as

$$\log f(\mathbf{v}'_1) + \log f(\mathbf{v}'_2) = \log f(\mathbf{v}_1) + \log f(\mathbf{v}_2) \quad (1.43)$$

which tells us that $\log f(\mathbf{v}_1) + \log f(\mathbf{v}_2)$ can only be a linear function of quantities that are conserved in a collision, *i.e.* the energy $\epsilon = \frac{1}{2}mv^2$ and the momentum $\mathbf{p} = m\mathbf{v}$ of the particles. It is straightforward to verify that $\log f = \log a + b\epsilon + \mathbf{c} \cdot \mathbf{v}$, for constant a, b, \mathbf{c} satisfies equation (1.43). Thus the distribution function has the form $f = a \exp(b\epsilon + \mathbf{c} \cdot \mathbf{v})$. We conclude that

$$f^0(\mathbf{v}) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[-\frac{m}{2k_B T} (\mathbf{v} - \hat{\mathbf{v}})^2 \right] \quad (1.44)$$

Boltzmann's H function¹⁵ is of fundamental importance since it provides a direct link with the entropy of a gas and from there a derivation of the second law of thermodynamics from micro-physics:

Relationship between Boltzmann's H and the entropy per unit volume S_v

$$S_v = -k_B H \quad (1.45)$$

Whereas S is defined only for systems in equilibrium, there is no such restriction on H .

¹⁴ Note that in taking this over to a relativistic situation the transition rate $d\sigma(\mathbf{v}_1, \mathbf{v}_2 \rightarrow \mathbf{v}'_1, \mathbf{v}'_2)$ would have to be Lorentz invariant.

¹⁵ Boltzmann himself used the letter E until he switched to H in 1895.

The H -theorem tells us that if the internal energy is kept constant, the entropy per unit volume will never decrease. For a thorough and lucid description of this see ter Haar (1955, 1995).

1.4.4 Equilibrium distribution

To make progress it is necessary to calculate J and then solve the Boltzmann equation. This is no simple matter even in the most idealised of cases, the Boltzmann equation is a nonlinear integro-differential equation. The first level of simplification is to assume that, to first order, the distribution function f is the Maxwell-Boltzmann distribution having a temperature T , so that there is an expansion $f = f^0 + f^1 + \dots$ where:

Maxwell Boltzmann equilibrium distribution

$$f^0(\mathbf{v}) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[-\frac{m}{2k_B T} (\mathbf{v} - \hat{\mathbf{v}})^2 \right] \quad (1.46)$$

Here $\hat{\mathbf{v}}$ is the fluid velocity and n is the scattering particle density. The distribution $f^0(\mathbf{v})$ makes the collision integral vanish and describes a rarefied gas. As before, m is the particle mass. With this, the Boltzmann equation and its collision term simplify and become tractable in the simplest of cases such as in the case of a gas having a component of light particles interacting with a component of much heavier particles.

What is important is what is learned from these simple approximations. The most important issue is the role of the collision mean free path λ_{mfp} in determining the validity of the resulting hydrodynamic equations. If the collisional process has total cross section σ and the gas has density n , then

$$\lambda_{mfp} \sim (n\sigma)^{-1}. \quad (1.47)$$

The condition for the validity of the fluid approximation derived from the Boltzmann equation is that the scale of the velocity and density gradients should be much greater than the collision mean free path:

$$L \sim n |\nabla n|^{-1} \gg \lambda_{mfp}, \quad \text{Kn} = \frac{\lambda_{mfp}}{L} \ll 1 \quad (1.48)$$

with a similar statement for the velocities and other relevant macroscopic properties of the gas. The dimensionless number Kn is called the *Knudsen number*. When the collision mean free path is long enough to violate this requirement we are in the regime of what is referred to as free molecular flow and the distribution function is determined by the collisionless Boltzmann equation. When $\text{Kn} \sim 1$ we have to interpolate between two regimes or solve the Boltzmann equation.

1.4.5 Radiation damping of cosmological perturbations

The $\text{Kn} \sim 1$ regime is precisely the circumstance we encounter when the Universe becomes neutral. As the ionisation level declines the photon mean free path increases rapidly and

this has the consequence that the photon viscosity acts on larger scales until the collisions become so infrequent that there is, in effect, no more viscosity and the photons travel freely to the observer. This is one of the central problems of cosmology: its solution determines what we see when we look at the angular structure of the microwave background radiation on this last scattering surface.

The first solution to this problem as proposed by Peebles and Yu (1970) in which they wrote separate equations for the radiation field and for the matter field. The radiation equation was a Boltzmann equation with a collision term describing the Compton scattering of the photons off the electrons. The key issue was to do the calculation including relativistic effects, and so it was necessary to write the Boltzmann equation in terms of Lorentz invariant quantities.

1.5 Approximations for the collision term

1.5.1 Chapman-Enskog approximation

The concept of *mean free path* had been introduced by Clausius around 1859 to investigate the transport properties in gases such as heat conduction, viscosity and thermal diffusion. Following on from Clausius' initiative both Maxwell and Boltzmann had used what might be termed "the mean free path theory" to investigate the Boltzmann equation. However, their treatments had deficiencies when it came to verification of the calculated values of transport coefficients with experimental measurement, particularly when it came to thermal diffusion.

The resolution came with the work of Chapman (1916a,b, 1918) and Enskog (1917) who independently tackled the problem of solving the Boltzmann equation via perturbation theory and without any assumptions about mean free paths. This was a remarkable *tour de force* in technically complex mathematical procedures involving integral equations. Here it is only necessary to give a flavour of the process and leave the details to texts such as (Huang, 1989, Ch. 6.).

We will need to recall that in the collisionless version of the Boltzmann equation we must have

$$\int \mathcal{D}f d\mathbf{v} = 0, \quad \int \mathbf{v} \mathcal{D}f d\mathbf{v} = 0, \quad \int |\mathbf{v}|^2 \mathcal{D}f d\mathbf{v} = 0 \quad (1.49)$$

in order to obtain the standard equations of continuity, momentum conservation and energy conservation. Therefore this constraint has to be respected even with the perturbed distribution functions. These equations will have to hold at all orders of perturbation theory.

The first step is to suppose that the gas is locally almost in equilibrium, so that the particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ departs only slightly from the Maxwell-Boltzmann distribution:

$$f = f^0 + f^1 + f^2 + \dots \quad (1.50)$$

This is to be understood as an expansion in terms of the ratio of the mean free path to the

scale on which the macroscopic fluid quantities are varying, *i.e.* small Knudsen number, \mathbf{Kn} (equation 1.48). At the outset this limits the range of applicability of the approximation, but, notwithstanding that, the approximations is of central importance in many fields.

The level of complexity is further increased by the need to impose conditions on the perturbation f^1, f^2, \dots of the distribution function. It is important to impose the *solvability conditions* that the moments of the perturbation to the distribution function are zero:

$$\int f^i d\mathbf{v} = 0, \quad \int \mathbf{v} f^i d\mathbf{v} = 0, \quad \int |\mathbf{v}|^2 f^i d\mathbf{v} = 0, \quad i = 0, 1, 2, \dots \quad (1.51)$$

In numerical simulations of fluid flow based on the Boltzmann equation such conditions appear as extra constraint equations that become part of the solution.

The collision term (1.39) involves squares of distribution functions evaluated for different parameter values, and so involves products like $f(\mathbf{v}_1)f(\mathbf{v}_2)$, which for brevity we shall denote by $f_1 f_2$. Each of these functions will have an expansion like (1.50) and so the series expansion of the collision term will look like

$$J(f_1 f_2) = J(f_1^0 f_2^0) + [J(f_1^0 f_2^1) + J(f_1^1 f_2^0)] + \dots = J^0 + J^1 + \dots \quad (1.52)$$

These perturbed collision terms must satisfy

$$\int J^i d\mathbf{v} = 0, \quad \int \mathbf{v} J^i d\mathbf{v} = 0, \quad \int |\mathbf{v}|^2 J^i d\mathbf{v} = 0, \quad i = 0, 1, 2, \dots \quad (1.53)$$

This requirement on the perturbed collision integral comes from equation (1.49). The requirement is automatically satisfied for the zero order equation for f^0 since f^0 is the equilibrium distribution for which $J(f_1^0 f_2^0) = 0$.

The perturbation f^1 is to be found from the integral equation

$$\mathcal{D}f^0 = J(f_1^0 f_2^1) + J(f_1^1 f_2^0) \equiv J^1 \quad (1.54)$$

Note that the left hand side involves only the zero-order distribution function, and because the departures from equilibrium are supposed small this term is itself of first order.

At this point everything is ready to compute f^1 and thence get the equations with their transport terms expressed explicitly in terms of macroscopic properties of the gas.

For a dilute monatomic gas of particles of mass m at temperature T the coefficients of thermal conductivity, κ , and viscosity, μ , are

$$\kappa = \frac{5\sqrt{\pi}}{8} \left(\frac{15}{4}\right) \frac{\sqrt{(k_B T/m)}}{\bar{\sigma}}, \quad \eta = \frac{5\sqrt{\pi}}{8} m \left(\frac{15}{4}\right) \frac{\sqrt{(k_B T/m)}}{\bar{\sigma}} \quad (1.55)$$

The quantity $\bar{\sigma}$ is an effective cross-section for the collisions and comes, after a lot of hard work, from evaluating the collision integral for monatomic gas particles. We learn from this that the viscosity and thermal conductivity vary as the square root of the temperature, which accords somewhat with the data.

The term $\bar{v} = \sqrt{(k_B T/m)}$ is the thermal velocity of the particles, while $\lambda = 1/(n\bar{\sigma})$ is the collision mean free path. So we can express these coefficients in terms of λ of the mean free path defined in this way:

$$\kappa = \frac{15}{4} n \lambda \bar{v}, \quad \mu = \rho \lambda \bar{v}, \quad \rho = nm. \quad (1.56)$$

The ratio of viscosity to thermal conductivity expressed dimensionlessly as $\mathbf{Pr} = \mu c_p / \kappa$, which is called the *Prandtl number*. c_p is the heat capacity at constant pressure. Calculating this for the parameter values (1.55) gives $\mathbf{Pr} = 2/3$ independent of temperature and density. For a monatomic gas like Argon this is quite reasonable over the entire range of temperatures 100 K – 2000 K.

1.5.2 BGK collision model

An alternative to dealing with the details of cross sections for various processes is to simply model the collision term as a rate of relaxation from a non-equilibrium distribution towards a Maxwellian equilibrium state (Bhatnagar et al., 1954). In this case the Boltzmann equation takes on the form

Bhatnagar-Gross-Krook (BGK) equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \mathbf{g} \cdot \nabla_{\mathbf{v}} f = \frac{f - f^M}{\tau} \equiv C(f) \quad (1.57)$$

where the equilibrium distribution f^M is the Maxwellian (1.46). The parameter τ is a relaxation timescale which is simply a phenomenological parameter which will be on the order of the collision time for the particles making up the fluid. When deviations from equilibrium are small the external force term is often expressed in terms of the equilibrium distribution f^0 and taken over to the right hand side:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f = \frac{f - f^M}{\tau} - \mathbf{g} \cdot \nabla_{\mathbf{v}} f^M \quad (1.58)$$

The BGK collision term $C(f)$ in (1.57) has to satisfy the physical conditions

$$\int C(f) d\mathbf{v} = 0, \quad \int \mathbf{v} C(f) d\mathbf{v} = 0, \quad \int |\mathbf{v}|^2 C(f) d\mathbf{v} = 0 \quad (1.59)$$

(see equation 1.49) and

$$\int (\log f) C(f) d\mathbf{v} \leq 0 \quad \text{with equality if and only if } f = f^M \quad (1.60)$$

for consistency with the original form of the Boltzmann equation, (1.25), with collision term (1.39)

Since the left hand side of (1.57) is the Liouville operator, it leads to the same gas dynamic equations as the full Boltzmann equation (see the derivation of equations (1.17), 1.22) and (1.24)) with the same caveat that the right hand side is required to obtain the anisotropic fluxes and transport coefficients. An important aspect of the BGK model is that the *H*-theorem still holds true.

1.5.3 Lattice Boltzmann equation

The *Lattice Boltzmann equation* is a numerical approach deriving from earlier work by Frisch et al. (1986) and Wolfram (1986) on Lattice Gas Automata. The Lattice Boltzmann

equation ('LBE') is an important grid-based mechanism for finding numerical solutions of the BGK equations (1.57).

The particles are located at the grid points of a mesh and move to neighbouring grid points. On a two dimensional square lattice it would be usual to use the eight directions from a node pointing along the four grid lines and the four diagonals. This is referred to as the *nine-speed LBE*, the ninth speed being the zero speed when the particle stays where it is. This provides a discretisation of both the position and the velocity space.

Schematically the numerical integration takes place in two phases: collisions and then particle moving (or vice versa). Let us consider a very simple method. First a collision step is taken in which the distribution function at grid direction i is updated according to

$$\bar{f}_i(\mathbf{x}, t + \delta_t) = f_i(\mathbf{x}, t) + \frac{1}{\tau}(f_i^{eq} - f_i) \quad (1.61)$$

Then, in the second phase, the *advection phase*, this value of the distribution is then assigned (or 'streamed') to the neighbouring lattice node that lies in direction \mathbf{e}_i at $\mathbf{x} + \mathbf{e}_i\delta_t$:

$$f_i(\mathbf{x} + \mathbf{e}_i\delta_t, t + \delta_t) = \bar{f}_i(\mathbf{x}, \delta_t) \quad (1.62)$$

The number of directions, \mathbf{e}_i depends on the geometry of lattice.

Of course, the algorithms used vary in complexity according to the particular problem. For a review of the variety of algorithms and a discussion of the limitations of LBE see, for example, Nourgaliev et al. (2003). Finally, it should be remarked that the grid does not have to be uniform, see Ubertini et al. (2003) for a discussion of one type of non-uniform grid.

1.5.4 The Sanders-Prendergast Beam Scheme

The *Beam Scheme* of Sanders and Prendergast (1974)¹⁶ was perhaps the earliest numerical Boltzmann-based scheme for gas-dynamic simulations and has been a source of inspiration for the implementation of Lattice Boltzmann based gas dynamics.

In this scheme the \mathbf{x} -space is divided up into cells. Each cell has a number of particles, each of mass m , which are assigned velocity components which, in equilibrium, would be drawn from a Maxwell-Boltzmann distribution having parameters appropriate to the local macroscopic fluid parameters: the density ρ , velocity \mathbf{u} and temperature T .

This can be achieved with as few as three particles moving along each axis of their cell. In each direction there is one central beam whose velocity u_i is the local macroscopic fluid velocity in the i -direction. This is referred to as the 'central beam'. It has two 'side beams' having velocities $u_i \pm \sqrt{k_B T/m}$, thereby crudely mimicking a gas having temperature T . At each time-step the particles are moved according to their velocities and eventually end up in the neighbouring cells (the time-step has to be small enough that they do not travel beyond the neighbouring cells - this is effectively the Courant condition). Following the particle-moving step the mean and variance of the particle velocities in the newly occupied cells are computed and replaced by a new set of Maxwell-Boltzmann distributed beams.

¹⁶ This seminal paper goes under the inauspicious title of "The Possible Relation of the 3-kiloparsec Arm to Explosions in the Galactic Nucleus": the method is described in an Appendix.

This replacement process is the relaxation process and ensures that at each time-step the local particle distribution is in equilibrium. Thus the effective relaxation time-scale is the time for particles to move between cells.

While the Lattice Boltzmann approaches are predicated on the particle distribution being close to Maxwellian, this is not the case with the Beam Scheme. This gives the Beam Scheme an advantage over Lattice Boltzmann in being able to deal with highly compressible gases. On the other hand, the Lattice Boltzmann approaches are designed to emulate the physically correct transport coefficients and so can give meaningful numbers when simulating real flows. The Beam Scheme is less quantitative in terms of fluid flow simulations, but, of course, in a situation where a cosmological gas flow is being simulated, the underlying physics is at best a naïve model and the Beam Scheme can be tuned to give good results.

1.6 Scattering of photons by electrons

1.6.1 Scattering photons by non-relativistic electrons

An important, and relatively simple, case is provided by the Boltzmann equation as a description of the scattering of photons off non-relativistic electrons. This requires a modification of the collision term in (1.39) for two reasons. Firstly, we have two interacting species and secondly photons obey Bose-Einstein statistics, not Maxwell-Boltzmann statistics.

The mutual thermodynamic equilibrium of the electrons gives an energy distribution

$$f_e(E) = \frac{n_e}{(2\pi m_e k_B T)^{3/2}} e^{-E/k_B T} \quad (1.63)$$

as in equation (1.44), where n_e is the number density of electrons and m_e is the electron mass.

The photon-electron scattering process we are considering is

$$e(E) + \gamma(\nu) \rightleftharpoons e(E') + \gamma(\nu') \quad (1.64)$$

During these collisions energy is transferred to the photons. However, because the photons are bosons the probability of scattering a photon from frequency ν' to ν does not depend simply on the number of photons $n(\nu)$ having frequency ν , there is an additional factor of $1 + n(\nu)$ coming into play¹⁷.

So, in a macroscopically homogeneous and isotropic gas, the Boltzmann equation for the photon distribution will be

$$\frac{\partial n(\nu)}{\partial t} = \int d^3 \mathbf{p} \int_{\Omega} \left[f_e(\mathbf{p}') n(\nu') (1 + n(\nu)) - f_e(\mathbf{p}) n(\nu) (1 + n(\nu')) \right] \frac{d\sigma}{d\Omega} d\Omega \quad (1.65)$$

Here we write the electron distribution f_e in terms of the electron momentum $\mathbf{p} = m_e \mathbf{v}$

¹⁷ Photons, being bosons, tend to occupy the same quantum state, in contrast to fermions which are excluded from occupying the same quantum state. A gas of fermions would have a factor $(1 - n(\nu))$ factor.

instead of its velocity (as in 1.39) or the energy (as in 1.63). From here, we can prove H-theorems for Bosons and Fermions (using the appropriate $1 - n(\nu)$ factor) and derive the Bose-Einstein and Fermi-Dirac distribution functions. This equation was first written down by Kompaneets (1957) and independently by Dreicer (1964).

1.6.2 The Kompaneets equation

We can make progress with this equation under the conditions that (a) the photon energy is very much smaller than the electron rest-mass energy, $h\nu \ll m_e c^2$, and (b) the changes in the energy of the photon during collisions with electrons is small, $\delta\nu = \nu' - \nu \ll \nu$. The first condition tells us that, in this approximation, in the rest frame of the electron there is no change in the photon energy. The differential cross-section $d\sigma/d\Omega$ is then expressed in terms of the Thomson scattering cross-section, σ_T , as

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi}(1 + \cos^2 \theta) \quad (1.66)$$

Under these conditions we can expand the distributions $f_e(E)$ and $n(\nu)$ in series with respect to the energy change $\delta\nu$ to $O(\delta\nu)^2$:

$$n(\nu') = n(\nu) + \Delta \frac{\partial n}{\partial \nu} + \frac{1}{2} \Delta^2 \frac{\partial^2 n}{\partial \nu^2} + \dots \quad (1.67)$$

$$f_e(E') = f_e(E) + \hbar\Delta \frac{\partial f_e}{\partial E} + \frac{1}{2} (\hbar\Delta)^2 \frac{\partial^2 f_e}{\partial E^2} + \dots \quad (1.68)$$

where $\hbar = h/2\pi$ and $\Delta = \nu' - \nu$. These are to be substituted in (1.65) to give a solution of the photon distribution $n(\nu, t)$. This is a tedious process which is presented, for example, in Peebles (1993, p.582 and p.603 *et seq.*) and in Rybicki and Lightman (1985, §7.6). The result is the diffusion-type equation:

$$\frac{1}{\sigma_T n_e c} \frac{\partial n(\nu)}{\partial t} = \frac{k_B T_e}{m_e c^2} \left[\nu^2 \frac{\partial^2 n}{\partial \nu^2} + 4\nu \frac{\partial n}{\partial \nu} \right] + \frac{h\nu}{m_e c^2} \left[4n(1+n) + (1+2n)\nu \frac{\partial n}{\partial \nu} \right] \quad (1.69)$$

where T_e is the equilibrium electron temperature. This equation was first written down by Kompaneets (1957) (who included terms additional for contributions from free-free emission by the electrons) and by Weymann (1965).

A considerable simplification is achieved with the substitution of dimensionless variables:

$$x = \frac{h\nu}{k_B T_e}, \quad dy = \frac{k_B T_e}{m_e c^2} \sigma_T n_e c dt \quad (1.70)$$

$$\frac{\partial n}{\partial y} = \frac{1}{x^2} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2 \right) \right] \quad (1.71)$$

This is due to Zeldovich and Sunyaev (1969) and so is referred to as the *Zel'dovich Sunyaev equation*. y is called the *Comptonization parameter*.

1.6.3 Variants on the Kompaneets equation

The Kompaneets equation plays an important role in many areas of physics and consequently there are many adaptations to different problems. The model (1.69) for the photon gas includes only low energy electron-photon scattering. There are important variants of this equation in the astrophysical context.

Firstly should note the relativistic version derived by Challinor and Lasenby (1998) which develops the solution of the Boltzmann equation to second order in $k_B T_e / m_e c^2$ which provides small but important corrections to interpreting the Sunyaev-Zel'dovich effect in hot galaxy clusters.

Additionally there is the important role played by bremsstrahlung (free-free) radiation from the electrons creating photons at low energies. For electron temperatures T_e higher than the photon temperature, the scattering process (1.64) diffuses photons to higher energies, thereby creating a non-Planckian spectrum. However, free-free radiation creates photons at low energies. These photons can fill up the non-Planckian curve generated by the Compton process, re-establishing a Planck distribution at T_e . This more general version of the Kompaneets equation was discussed at length by Illarionov and Sunyaev (1974) and by Chan and Jones (1975a,b,c) who presented numerical simulations of the evolution of the spectrum. See Section ?? for a discussion on the cosmological context.

1.7 Relativistic Boltzmann equation

1.7.1 Relativistic Boltzmann models

The first step towards the relativistic Boltzmann equation was made by Jüttner in 1911 in a paper deriving the relativistic analogue of the Maxwell distribution. That might be said to be the first paper on relativistic kinetic theory but little more was done until the seminal work of Chernikov (1964a,b), Lindquist (1966) and Marle (1969).

The relativistic version of the Boltzmann equation is of importance in cosmology, and in the study of massive stars and relativistic accretion disks where radiation pressure and radiative transport play a role. Inevitably, relativistic kinetic theory carries with it a greater degree of technical complexity. So here we shall just present an outline of what is involved in going relativistic.¹⁸

The main technical issue here is to build a relationship between the position and momentum 3-spaces with which we are familiar and the locally Minkowski 4 + 4-space of 4-position and 4-momentum: (x^a, p^a) .¹⁹

¹⁸ The relativistic Boltzmann equation is a vast and technically complex subject that impinges on many branches of physics. The following are among a number of texts that I found particularly informative, even though at times highly technical: Weinberg (1972, §§2.10, 2.11), Liboff (1990), Cercignani (2002), Dodelson (2003), and Weinberg (2008, Appendices G and H).

¹⁹ The subject of 3-vectors and 4-vectors is covered in section ?. It is important to remember that we are working

1.7.2 Relativistic phase space

In the relativistic situation it is convenient to think in terms of the 4-momentum $p^a = (p^0, \mathbf{p})$ rather than the velocity \mathbf{v} of the particles in the gas. Thus the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ is replaced by the function $f(x^a, p^a)$. However, need to maintain a close link between the 4 + 4-dimensional relativistic version of the Boltzmann equation and the 3 + 3 dimensional physics of the gas as seen in our local inertial frame and in which we understand the Liouville theorem. We need to establish the physical connection between the 3 + 3 (\mathbf{x}, \mathbf{p}) subspace and its parent 4 + 4 space (x^a, p^a) .

The 4-momentum p^a for a particle of mass m is

$$p^a = (p^0, \mathbf{p}), \quad p^a p_a = -1, \quad p^0 = \sqrt{|\mathbf{p}|^2 + m^2 c^2} \quad (1.72)$$

With this we can show the important result that if (x'^a, p'^a) is the result of a Lorentz transform from (x^a, p^a) , then

Lorentz transformations of elemental 3-space volumes

$$p'^0 d^3 \mathbf{x}' = p^0 d^3 \mathbf{x}, \quad \frac{d^3 \mathbf{p}'}{p'^0} = \frac{d^3 \mathbf{p}}{p^0} \quad (1.73)$$

$$d^3 \mathbf{x}' d^3 \mathbf{p}' = d^3 \mathbf{x} d^3 \mathbf{p} \quad (1.74)$$

The first of these equations (1.73) shows that the invariant element of volume for momentum 3-space integration is $d^3 \mathbf{p} / p^0$. The second equation (1.74) shows that the 3 + 3 element of phase space volume is Lorentz invariant and so $f(x^a, p^a) d^3 \mathbf{x} d^3 \mathbf{p}$ is also Lorentz invariant if $f(x^a, p^a)$ itself is expressed in terms of Lorentz invariant quantities. This provides our entry into relativistic phase space.

1.7.3 Lorentz transformation of phase space volumes

We now prove the result 1.74 describing the invariance of the phase space volume under Lorentz transformations in a simple case (see also Liboff (1990, §6.2.6) or Treumann et al. (2011) for other derivations).

We write the 4-momentum p^a for a particle of mass m as

$$p^a = (p^0, \mathbf{p}), \quad p^a p_a = -1, \quad p^0 = \sqrt{|\mathbf{p}|^2 + m^2 c^2} \quad (1.75)$$

With this we can show the important result that if (x'^a, p'^a) is the result of a Lorentz trans-

with a $(-+++)$ metric signature. (Using the alternative $(+---)$ signature changes the signs of terms appearing in some of the equations. This is summarised in the Preface, page ?? *et seq.*).

form from (x^a, p^a) , then

$$p'^0 d^3 \mathbf{x}' = p^0 d^3 \mathbf{x}, \quad \frac{d^3 \mathbf{p}'}{p'^0} = \frac{d^3 \mathbf{p}}{p^0} \quad (1.76)$$

$$d^3 \mathbf{x}' d^3 \mathbf{p}' = d^3 \mathbf{x} d^3 \mathbf{p} \quad (1.77)$$

We transform the 4-momentum (p^0, \mathbf{p}) to another inertial frame via a Lorentz transformation between two frames moving relative to one another at speed v along their common x^1 -axis. In the second inertial frame the 4-momentum is:

$$\begin{pmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}, \quad \beta = v/c, \quad \gamma = (1 - \beta^2)^{-1/2}, \quad (1.78)$$

(see equation ??), from which we get

$$p'^0 = \gamma p^0 - \gamma\beta p^1, \quad p'^1 = \gamma\beta p^0 + \gamma p^1. \quad (1.79)$$

We can compute the change in the volume element $d^3 \mathbf{p}$ using the Jacobian of the transformation $\mathbf{p} \rightarrow \mathbf{p}'$:²⁰

$$\frac{\partial(p'^1 p'^2 p'^3)}{\partial(p^1 p^2 p^3)} = \begin{vmatrix} \gamma \left(1 - \beta \frac{\partial p^0}{\partial p^1}\right) & -\beta\gamma \frac{\partial p^0}{\partial p^2} & -\beta\gamma \frac{\partial p^0}{\partial p^3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \gamma \left(1 - \beta \frac{\partial p^0}{\partial p^1}\right). \quad (1.80)$$

From the last relationship in equations (1.75) we have $\frac{\partial p^0}{\partial p^1} = \frac{p^1}{p^0}$, whence

$$\frac{\partial(p'^1 p'^2 p'^3)}{\partial(p^1 p^2 p^3)} = \gamma \left(1 - \beta \frac{p^1}{p^0}\right) = \frac{p'^0}{p^0}. \quad (1.81)$$

on using equation (1.79). This is the factor by which the elemental 3-volume changes in the transformation $d^3 \mathbf{p} \rightarrow d^3 \mathbf{p}'$, so

$$\frac{d^3 \mathbf{p}'}{p'^0} = \frac{d^3 \mathbf{p}}{p^0} \quad (1.82)$$

A similar argument for the transformation of the element 3-volume $d^3 \mathbf{x} \rightarrow d^3 \mathbf{x}'$ gives

$$\mathbf{p}'^0 d^3 \mathbf{x}' = \mathbf{p}^0 d^3 \mathbf{x} \quad (1.83)$$

and so finally we have our result that

$$d^3 \mathbf{x}' d^3 \mathbf{p}' = d^3 \mathbf{x} d^3 \mathbf{p}. \quad (1.84)$$

The elemental volume of the 3 + 3 phase space is invariant under Lorentz transformation.

1.7.4 Relativistic Boltzmann equation

This is in the
wrong place

The general relativistic Boltzmann equation can be written as

$$\left[p^a \frac{\partial}{\partial x^a} + \Gamma_{bc}^a p^b p^c \frac{\partial}{\partial p^a} \right] f = \left(\frac{\partial f}{\partial \tau} \right)_{coll} \quad (1.85)$$

where Γ_{bc}^a is the standard Christoffel symbol accounting for the gravitational force field, and $(\partial f / \partial \tau)_{coll}$ is a relativistic collision term (Lindquist, 1966).²¹

The simplest form for the metric of a perturbed FRW universe requires that we introduce two fluctuating potentials $\Psi(\mathbf{x}, t)$, corresponding to the Newtonian weak field potential, and $\Phi(\mathbf{x}, t)$, describing the curvature fluctuations. The curvature contribution acts on the spatial part of the metric:

$$ds^2 = -(1 + 2\Psi) dt^2 + a^2(t)(1 + 2\Phi)(dr^2 + r^2 d\Omega^2) \quad (1.86)$$

1.7.5 Relativistic Maxwell-Boltzmann-Jüttner distribution

This is a bit
fragmentary -
reorganise

We need to use the distribution function to define relativistic quantities in the local fluid rest frames, then we are defining physical quantities that are direct analogues of the non-relativistic quantities. This means that our interest is in the number of particles in a volume $d^3\mathbf{x} d^3\mathbf{p}$ of the (x^a, p^a) phase space.

The equilibrium Maxwell-Boltzmann distribution derived by Jüttner (1911) is

$$f^{eq} = Z^{-1} \exp(-p^a u_a / k_B T) \quad (1.87)$$

where Z is a normalisation constant. The 4-velocity is $u^a = \gamma(u)(c, \mathbf{u})$, where \mathbf{u} is the 3-velocity of the fluid in the chosen frame of reference. The distribution function f^{eq} is referred to as the *Maxwell-Jüttner distribution*.

The goal of relativistic kinetic theory is to derive a set of Lorentz invariant equations describing fluid flow, and to use, insofar as possible, the same physical variables that in the limit of slow motions are the descriptive variables of the fluid state with which we are familiar.

1.7.6 Energy momentum tensor: dissipationless

This all uses $c = 1$

- fix it The energy momentum tensor for a perfect fluid is

$$T_{eq}^{ab} = \mu u^a u^b + P h^{ab}, \quad h^{ab} = g^{ab} + u^a u^b \quad (1.88)$$

We write this with the space-time metric g^{ab} in accord with the Principle of Covariance which tells us to replace the Minkowski metric η^{ab} with the Riemannian metric g^{ab} . The special relativistic version, discussed in section ??, can be traced back to the work of von Laue (1911). So to get this expression does not in fact require the relativistic Boltzmann

Check cross
reference to SR
version

²⁰ The Jacobian $\frac{\partial(p'^1 p'^2 p'^3)}{\partial(p^1 p^2 p^3)}$ is the 3×3 determinant with elements $\frac{\partial p'^i}{\partial p^j}$.

²¹ Although Γ_{bc}^a is not a tensor, the left hand side of this equation is nevertheless coordinate invariant.

equation. We need that if we are to get the correct form of the energy moment tensor including energy transport terms. We come to that shortly.

As explained in Section ??, μ is the (rest-frame) energy density of the fluid and P is the isotropic pressure (written here in upper-case so as to avoid confusion with the magnitude of the momentum 4-vector p^a). μ is related to the local rest mass density ρ and the specific internal energy of the fluid, ϵ by

$$\mu = \rho(1 + \epsilon) \quad (1.89)$$

The energy associated with internal motions contributes to ϵ . The ρ , μ , and P are local rest-frame values and so correspond to the quantities we know from classical gases. Note that $\mu = u_a u_b T^{ab}$

We shall define two tensors that are moments of the distribution function $f(x^a, p_b)$ with respect to the 4-momenta p^a . The first is the particle 4-flux, $N^a(\mathbf{x})$:

$$N^a(\mathbf{x}) = \int p^b f(x^a, p^b) \frac{d^3 \mathbf{p}}{p^0} \quad (1.90)$$

Recall equation (1.74) that $d^3 \mathbf{p}/p^0$ is the invariant volume element in momentum space. This will be used to define what is meant by the mean flow velocity in an element of spatial volume.

The energy momentum tensor can be recovered from the distribution function by the equation

$$T^{ab} = \int p^a p^b f(x^a, p^b) \frac{d^3 \mathbf{p}}{p^0}, \quad p^a p_a = m^2 \quad (1.91)$$

Using the Maxwell-Jüttner equilibrium distribution in this integral provides the energy momentum tensor for the relativistic ideal gas in the familiar form (1.88).

1.7.7 Energy momentum tensor: with dissipation

Small deviations from the equilibrium distribution result in a slightly different energy momentum tensor $T^{ab} = T_{eq}^{ab} + \Pi^{ab}$, where the perturbation Π^{ab} provides the fluxes and transport coefficients for the more general equations that include dissipative processes such as thermal conduction and viscosity.

There is some intuitive guidance as to what Π^{ab} might look like in the case of a fluid with non-zero thermal conductivity and shear viscosity. From work going back to the mid 19th century by Stokes and others, we know that the thermal conductivity results in a transfer of energy via a flux of energy: it is a vector quantity much like the Poynting vector \mathbf{S} of electromagnetism. Likewise we know that the viscosity of the fluid is represented by a symmetric and trace-free fluid stress tensor that depends on the velocity gradients. This indicates that we need to add terms involving a vector quantity q^a and a tensor quantity π^{ab} into the energy momentum tensor in such a way that they appear as depicted in Figure ?. So π^{ab} has to act only in the spatial part of the energy momentum tensor.

The solution to writing down the energy momentum tensor of a dissipative fluid is thus

provided by the fact that any symmetric tensor T^{ab} can be decomposed into

$$T^{ab} = \mu u^a u^b + P h^{ab} + 2u^{(a} q^{b)} + \pi^{ab} \quad (1.92)$$

$$h^{ab} = g^{ab} + u^a u^b, \quad u^a u_a = -1, \quad (1.93)$$

$$u_a q^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi^a_a = 0. \quad (1.94)$$

Here we are using the convention that pairs of indices enclosed in round brackets are symmetrised: $2u^{(a} q^{b)} = u^a q^b + u^b q^a$.

If we write equation (1.92) in a local inertial frame where $u^a = (c, \mathbf{0})$ and look back at Figure ?? we see that the term $2u^{(a} q^{b)}$ fills in the entries marked (T^{01}, T^{02}, T^{03}) and (T^{10}, T^{20}, T^{30}) , which are identified as the momentum flux in the figure, and, because of the condition $\pi_{ab} u^b = 0$, the tensor π^{ab} is zero everywhere except for the spatial entries $\pi^{\mu\nu}$, $\mu, \nu = 1, 2, 3$, *i.e.* the shear stresses. In that inertial frame the pressure fills in the entries (T^{11}, T^{22}, T^{33}) and so is the isotropic pressure.

1.7.8 Defining the macroscopic 4-velocity u^a

There is a technical issue that arises at this point: what is u^a in these equations? u^a has been referred to as the mean localised fluid flow velocity. However, away from the equilibrium that concept becomes somewhat more complicated, particularly in the relativistic regime where, for example, heat flux carries energy and hence makes a contribution to the momentum. To appreciate what difference being away from equilibrium may make we can think of the transport processes as being mediated by a separate fluid having a different velocity from the local mean gas velocity: in a gas made of photons and free electrons this is in fact a good model. If the two fluids had different velocities even on the ‘local’ scale, what would we mean by the ‘fluid velocity’?

The question arises as to what is it, at the microscopic level, that determines the 4-velocity u^a ? There are two approaches to this question, one due to Landau (1949, §127) and the other due to Eckart (1940). Landau takes the velocity u^a to be the fluid velocity in the frame where the momentum of the fluid element is zero, while Eckart uses the frame in which the local particle flow velocity is zero.

Both variants are in current use, though the Landau version of the energy momentum tensor tends to be favoured for studies involving the relativistic Boltzmann equation.

The situation can be formalised as follows. The 4-flux of particles is given in terms of the particle distribution function by

$$N^a = \int p^a f dp \quad (1.95)$$

where dp is the Lorentz invariant element of volume in momentum space: $dp = d^3 p / p_0$, as explained in equations (1.72, 1.73).

The Eckart energy momentum tensor is

$$T_E^{ab} = \mu u^a u^b + P^{ab} + u^a q^b + u^b q^a \quad (1.96)$$

where q^a is the *heat flux* 4-vector and P^{ab} is the pressure tensor, which need not be isotropic in the dissipative case.

1.7.9 Relativistic BGK equation

The relative simplicity of the BGK model equation (1.57) with no external forces suggests that it might be written in a relativistic form:

$$\partial_a(p^a f) = \frac{p^a u_a}{\tau}(f - f^0) \quad (1.97)$$

References

- Arad, I., and Johansson, P. H. 2005. A numerical comparison of theories of violent relaxation. *MNRAS*, **362**(Sept.), 252–262.
- Arad, I., and Lynden-Bell, D. 2005. Inconsistency in theories of violent relaxation. *MNRAS*, **361**(Aug.), 385–395.
- Arnold, V.I. 1980. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics, vol. Vol. 60. Springer.
- Baxter, G. W., and Olafsen, J. S. 2007. Experimental Evidence for Molecular Chaos in Granular Gases. *Physical Review Letters*, **99**(2), 028001.
- Bertschinger, E. 1995. Cosmological dynamics. *NASA STI/Recon Technical Report N*, **96**(Jan.), 22249. This lecture course is also available electronically as <http://arxiv.org/pdf/astro-ph/9503125v1.pdf> and on-line at http://ned.ipac.caltech.edu/level5/March02/Bertschinger/Bert_contents.html.
- Bhatnagar, P. L., Gross, E. P., and Krook, M. 1954. A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems. *Physical Review*, **94**(May), 511–525.
- Binney, J., and Tremaine, S. 2008. *Galactic Dynamics*. 2nd. edn. Princeton Series in Astrophysics. Princeton University Press.
- Boltzmann, L. 1872. Weitere Studien über das Wärmegleichgewicht unter Gasmoleculen. *Wien Ber*, **66**, 275.
- Brush, S.G. 1961. John James Waterston and the Kinetic Theory of Gases. *American Scientist*, **49**, 202–214.
- Brush, S.G., and Hall, N.S. 2003. *The Kinetic Theory of gases: An Anthology of Classic Papers with Historical Commentary*. History of Modern Physical Sciences - Vol. 1. Imperial College Press.
- Cercignani, C. 2002. *The Relativistic Boltzmann Equation: Theory and Applications*. Birkhäuser Basel.
- Challinor, A., and Lasenby, A. 1998. Relativistic Corrections to the Sunyaev-Zeldovich Effect. *ApJ*, **499**(May), 1.
- Chan, K. L., and Jones, B. J. T. 1975a. Distortions of the 3 K background radiation spectrum - Observational constraints on the early thermal history of the universe. *ApJ*, **195**(Jan.), 1–11.
- Chan, K. L., and Jones, B. J. T. 1975b. Distortions of the microwave background radiation spectrum in the submillimeter wavelength region. *ApJ*, **198**(June), 245–248.
- Chan, K. L., and Jones, B. J. T. 1975c. The Evolution of the Cosmic Radiation Spectrum Under the Influence of Turbulent Heating. II. Numerical Calculation and Application. *ApJ*, **200**(Sept.), 461–470.

- Chapman, S. 1916a. On the Law of Distribution of Molecular Velocities, and on the Theory of Viscosity and Thermal Conduction, in a Non-Uniform Simple Monatomic Gas. *Royal Society of London Philosophical Transactions Series A*, **216**, 279–348.
- Chapman, S. 1916b. The Kinetic Theory of Simple and Composite Monatomic Gases: Viscosity, Thermal Conduction, and Diffusion. *Royal Society of London Proceedings Series A*, **93**(Dec.), 1–20.
- Chapman, S. 1918. On the Kinetic Theory of a Gas. Part II: A Composite Monatomic Gas: Diffusion, Viscosity, and Thermal Conduction. *Royal Society of London Philosophical Transactions Series A*, **217**, 115–197.
- Chapman, S., and Cowling, T.G. 1939. *The Mathematical Theory of Non-uniform Gases*. 1st. edn. Cambridge University Press. 2nd. edition 1952; 3rd. edition 1991.
- Chernikov, N.A. 1964a. Microscopic Foundation of Relativistic Hydrodynamics. *Acta Physica Polonica*, **27**, 465.
- Chernikov, N.A. 1964b. The relativistic gas in the gravitational field. *Acta Physica Polonica*, **23**, 629.
- Dodelson, S. 2003. *Modern cosmology*. Academic Press.
- Dreicer, H. 1964. Kinetic Theory of an Electron-Photon Gas. *Physics of Fluids*, **7**(May), 735–753.
- Eckart, C. 1940. The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid. *Physical Review*, **58**(Nov.), 919–924.
- Enskog, D. 1917. *Kinetische Theorie der Vorgänge in mässig verdünnten Gasen*. Almqvist and Wiksell, Uppsala.
- Euler, L. 1757. Principes generaux du mouvement des fluides. *Mémoires de l'académie des sciences de Berlin*, **11**, 274–315.
- Frisch, U., Hasslacher, B., and Pomeau, Y. 1986. Lattice-gas automata for the Navier-Stokes equation. *Physical Review Letters*, **56**(Apr.), 1505–1508.
- Gull, S.J. 1989. *Some Misconceptions about Entropy*. Possibly volatile URL. [Online]. Available: <http://www.ucl.ac.uk/ucesjph/reality/entropy/text.html>.
- Huang, K. 1989. *Statistical Mechanics*. Wiley; 2nd. edition.
- Illarionov, A. F., and Sunyaev, R. A. 1974. Comptonization, the spectrum of RELICT radiation, and the thermal history of the universe. *AZh*, **51**(Dec.), 1162–1176.
- Jaynes, E. T. 1965. Gibbs vs Boltzmann Entropies. *American Journal of Physics*, **33**(May), 391–398.
- Jeans, J. H. 1915. On the theory of star-streaming and the structure of the universe. *MNRAS*, **76**(Dec.), 70–84.
- Jüttner, F. 1911. Das Maxwellsche Gesetz der Geschwindigkeits - verteilung in der Relativtheorie. *Ann Physik (Leipzig)*, **339**, 856–882.
- Kompaneets, A.S. 1957. The Establishment of Thermal Equilibrium between Quanta and Electrons. *Sov. Phys. - JETP*, **4**, 730–.
- Liboff, R.L. 1990. *Kinetic Theory: Classical, Quantum, and Relativistic Description*. 3rd. edn. Graduate Texts in Contemporary Physics. Springer.
- Lindquist, R. W. 1966. Relativistic transport theory. *Annals of Physics*, **37**(May), 487–518.

- Loschmidt, J. 1876. Über den Zustand des Wärmegleichgewichtes eines Systems von Körpern mit Rücksicht auf die Schwerkraft, I. *Sitzungsber. Kais. Akad. Wiss. Wien, Math. Naturwiss. Classe, II*, **73**, 128–142.
- Lynden-Bell, D. 1962a. Stellar dynamics. Only isolating integrals should be used in Jeans theorem. *MNRAS*, **124**, 1.
- Lynden-Bell, D. 1962b. Stellar dynamics. Potentials with isolating integrals. *MNRAS*, **124**, 95.
- Marle, C. 1969. Sur l'établissement des équations de l'hydrodynamique des fluides relativistes dissipatifs. I. - l'équation de Boltzmann relativiste. *Ann. Inst. H. Poincaré*, **10**, 67–126.
- McNamara, G. R., and Zanetti, G. 1988. Use of the Boltzmann equation to simulate lattice-gas automata. *Physical Review Letters*, **61**(Nov.), 2332–2335.
- Newton, I. 1726. *The Principia: Mathematical Principles of Natural Philosophy (3rd edition)*. Royal Society of London.
- Nourgaliev, R.R., Dinh, T.N., Theofanous, T.G., and Joseph, D. 2003. The lattice Boltzmann equation method: theoretical interpretation, numerics and implications. *Int. J. Multiphase Flow*, **29**, 117–169.
- Peebles, P. J. E. 1993. *Principles of physical cosmology*. Princeton University Press.
- Peebles, P. J. E., and Yu, J. T. 1970. Primeval Adiabatic Perturbation in an Expanding Universe. *ApJ*, **162**(Dec.), 815–+.
- Poincaré, H. 1911. *Leçons sur les hypothèses cosmogoniques professées à la Sorbonne*. 1st. edn. Librairie Scientifique. Paris, A. Hermann et fils.
- Rybicki, B.B., and Lightman, A.P. 1985. *Radiative Processes in Astrophysics*. Wiley VCH.
- Sanders, R. H., and Prendergast, K. H. 1974. The Possible Relation of the 3-KILOPARSEC Arm to Explosions in the Galactic Nucleus. *ApJ*, **188**(Mar.), 489–500.
- ter Haar, D. 1955. Foundations of Statistical Mechanics. *Re. Mod. Phys.*, **27**(3), 279–338.
- ter Haar, D. 1995. *Elements of Statistical Mechanics*. Butterworth-Heinemann.
- Treumann, R. A., Nakamura, R., and Baumjohann, W. 2011. Relativistic transformation of phase-space distributions. *Annales Geophysicae*, **29**(July), 1259–1265.
- Ubertini, S., Bella, G., and Succi, S. 2003. Lattice Boltzmann method on unstructured grids: Further developments. *Phys. Rev. E*, **68**(1), 016701.
- von Laue, M. 1911. Zur Dynamik der Relativitätstheorie. *Annalen der Physik*, **35**, 524–542.
- Waterston, J.J. 1892. On the physics of media that are composed of free and perfectly elastic molecules in a state of motion. *Phil. Trans. Roy. Soc. A*, **183**, 1–79.
- Watson, K. M. 1956. Use of the Boltzmann Equation for the Study of Ionized Gases of Low Density. I. *Physical Review*, **102**(Apr.), 12–19.
- Weinberg, S. 1972. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*.
- Weinberg, S. 2008. *Cosmology*. Oxford University Press.
- Weymann, R. 1965. Diffusion Approximation for a Photon Gas Interacting with a Plasma via the Compton Effect. *Physics of Fluids*, **8**(Nov.), 2112–2114.
- Wolfram, S. 1986. Cellular automaton fluids 1: Basic theory. *Journal of Statistical Physics*, **45**(Nov.), 471–526.

- Zeldovich, Y. B., and Sunyaev, R. A. 1969. The Interaction of Matter and Radiation in a Hot-Model Universe. *Ap&SS*, **4**(July), 301–316.