Hilbert's Stress-Energy Tensor

A Supplement to "Precision Cosmology"

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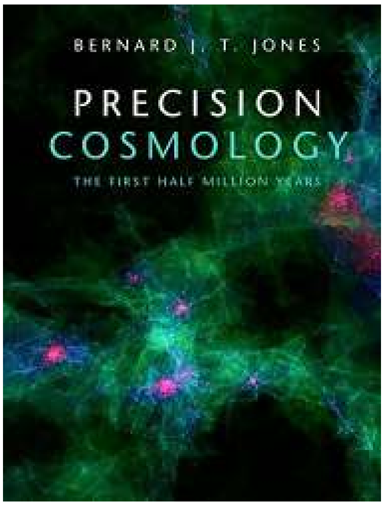

Hilbert was interested in unifying the forces of Nature. His unifying principle was that electromagnetism was the basis of all forces. The most important development to emerge from this was an expression for the stress energy tensor.

This is one of a set of Supplementary Notes and Chapters to "Precision Cosmology".
Some of these Supplements might have been a chapter in the book itself, but were regarded either as being somewhat more specialised than the material elsewhere in the book, or somewhat tangential to the main subject matter.
The are mostly early drafts and have not been fully proof-read.
Please send comments on errors or ambiguities to "`PrecisionCosmology(at)gmail.com`".

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1.1 Hilbert stress-energy tensor

In general relativity the action is split into the sun of two terms: a gravitational term with action $\mathcal{L}_{GR}$ and a matter term with action $\mathcal{L}_M$:

$$\mathcal{S} = \mathcal{S}_{GR} + \mathcal{S}_M, \quad \mathcal{S}_{GR} = \int_{\mathcal{D}} R\sqrt{-g}d^4x, \quad \mathcal{S}_M = \int_{\mathcal{D}} L_M\sqrt{-g}d^4x. \tag{1.1}$$

Consider first the gravitational component $\mathcal{S}_{GR}$. If we write $R = g^{ab}R_{ab}$ we have three quantities under the integral sign to vary: R_{ab}, $\sqrt{-g}$ and g_{ab}. Both R_{ab} and $\sqrt{-g}$ can be expressed in terms of g^{ab} and then varied (*i.e.* 'differentiated') with respect to g_{ab}. Deriving this is a tedious process the details of which can be found in the detailed descriptions of Carroll (2003, §4.3) and of Hobson et al. (2006, §19.8). The result is

$$\delta\mathcal{S}_{GR} = \int_{\mathcal{D}} (R_{ab} - \frac{1}{2}Rg_{ab})\delta g^{ab}\sqrt{-g}d^4x \tag{1.2}$$

which yields the vacuum Einstein equations.

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{S}_{GR}}{\delta g_{ab}} = 0 \Rightarrow R_{ab} - \frac{1}{2}Rg_{ab} = 0 \tag{1.3}$$

since $\delta\mathcal{S}_{GR} = 0$ in () for all variations δg^{ab} of the metric.

Since we know that in the presence of matter, the right hand side of the Einstein equations is the energy momentum tensor, it stands to reason that

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{S}_M}{\delta g_{ab}} = \kappa T_{ab}, \tag{1.4}$$

the energy momentum tensor of the matter. This is equivalent to

$$T^{ab} = 2\frac{\delta L_M}{\delta g_{ab}} \tag{1.5}$$

This gives us a powerful method of getting a relativistically correct symmetric energy momentum tensor without recourse to Noether's theorem. If we know the classical Lagrangian for the field we simply have to replace all ordinary derivatives with the corresponding covariant derivatives.

This result is somewhat surprising since there can be no analogy of this outside of the framework of general relativity. We appear to have calculated the energy momentum tensor by using an entity, the metric, that we would not normally think of as an essential part of our physical experience.

The energy momentum tensor of the matter component is

$$T^{ab} = \frac{2}{\sqrt{|g|}} \frac{\delta \sqrt{|g|}\mathcal{L}_M}{\delta g_{ab}} = 2\frac{\delta \mathcal{L}_M}{\delta g_{ab}} + g^{ab}\mathcal{L}_M \tag{1.6}$$

This is the *Hilbert stress energy tensor* which is guaranteed to be both symmetric in its indices and gauge invariant. Hence it does not suffer the disadvantages of the energy momentum tensor derived via Noether's theorem. This somewhat remarkable result is discussed in depth in Hawking and Ellis (1975, Section 3.3).

There is a discussion of the relationship between the Belinfante-Rosenfeld energy momentum tensor in Gotay and Marsden (1992); Leclerc (2006).

1.2 Positive Energy conditions

The energy momentum tensor for pressure free matter ("dust") is

$$T^{ab} = \rho u^a u^b, \quad u^a u_a = -1 \tag{1.7}$$

where u^a is the four-velocity. u^a is also the normalised tangent vector to the observer's world line. This tells us that

$$\rho = T_{ab} u^a u^b \tag{1.8}$$

For this quantity to be strictly positive we have $T^{ab} u_a u_b > 0$.

The *weak energy energy condition* is that an energy momentum tensor should satisfy the positivity constraint

Weak energy condition

$$T^{ab} V_a V_b \geq 0, \quad V_a \text{ timelike} \tag{1.9}$$

for all timelike vectors V_a

This asserts that the energy density measured by any observer is always non-negative. The *strong energy condition* asserts a stronger lower bound on $T^{ab} V_a V_b$ that

$$T^{ab} V_a V_b > -\tfrac{1}{2} T \tag{1.10}$$

for all time-like vectors V_a. The alternative way of writing this condition is There is a stricter condition :

Strong energy condition

$$(T^{ab} - \tfrac{1}{2} g^{ab} T) V_a V_b > 0 \quad \text{or} \quad R^{ab} V_a V_b > 0 \tag{1.11}$$

for all timelike vectors V_a.

from equation (**??**). This, in effect, puts a stronger limit on the pressure than the weak energy condition.

There is a stricter condition :

Dominant energy condition

$$T^{ab}W_aW_b \geq 0 \quad \text{and} T^{ab}W_a \text{ is non-spacelike} \tag{1.12}$$

for all timelike W_a

The term $T^{ab}W_a$ is the energy-momentum 4-current density measured by an observer moving with 4-velocity W^a. Imposing the non-space-like restriction on this flux effectively restricts the speed of the energy flow to being less than the speed of light. The dominant energy condition asserts that the the pressure should not exceed the energy density, which is the case for all known forms of matter.

Hawking and Ellis (1975, §4.3) have an extensive discussion on such energy conditions and their relationship to the inevitability of various types of singularity in space-time.

1.3 Quantum vacuum

In Quantum Field Theories (QFT) the vacuum state is the lowest energy state of a collection of quantum fields: the quantum vacuum is everywhere dominated by what are referred to as “zero point fluctuations” in energy which endow the vacuum with a non-zero energy density. These fluctuations are described as being due to the appearance and disappearance of virual particles in the vacuum. The origin of this idea in its modern form traces back to Dirac (1927).

The verification of this remarkable view came in 1997 with the measurement an effect that had been predicted within the framework of Quantum Electrodynamics in 1948 by Casimir, not surprsingly called *Casimir Effect* (Casimir, 1948; Casimir and Polder, 1948)).[1] See Rugh et al. (1999); Rugh and Zinkernagel (2002) for a historical and cosmological perspective on the quantum vacuum.

In view of the fact that the quantum vacuum does not have zero energy we should draw distinction between a physical (*i.e.* quantum) vacuum solution of the Einstein equations and the classical empty space solution where $T_{ab} = 0$. What should replace the energy momentum tensor T_{ab} for the vacuum is its quantum expectation:

$$R_{ab} - \tfrac{1}{2}g_{ab}R - \Lambda g^{ab} = \kappa\langle \hat{T}_{ab}\rangle \tag{1.13}$$

where the right hand side is the expectation value of the energy momentum tensor, now viewed as a quantum operator for the vacuum state. The question is, of course, how one might calculate the object on the right hand side.

The fact that the value of Λ derived from quantum field theory is so outrageously large

[1] There are two forms of the Casimir Effect: the classical effect in which the finite vacuum energy causes an attraction between two closely placed metal plates and the Dynamic Casimir Effect in which light is created out of the vacuum by ultra-fast moving mirrors. The dynamical Casimir effect was experimentally verified in 2011.

in comparison to the observed value remains one of the fundamental mysteries of modern physics: but we are not going to throw away the highly successful "Standard Model" over this problem. An interesting discussion of this problem is provided by the Nobel prizewinner Gerard 't Hooft (2001).[2]

[2] Zel'dovich suggested a piece of numerology, using only the fundamental constants of nature, that gets closer to the value of Λ:

$$\Lambda \sim \frac{G^2 m_p^6}{\hbar^4} \tag{1.14}$$

without any indication of how this may come about. However, it is only 7 orders of magnitude larger than the required value. As Zel'dovich himself points out, one get get even closer replacing m_p^6 by $m_p^4 m_e^2$.

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